Sector-level Attribution Effects
with
Compounded Notional Portfolios

Why Would We Want Them
and
How Can We Get Them?

Mark R. David, CFA
The Setup - What is Arithmetic Time Period Linking Trying to Accomplish?

➢ Additivity
  • of sectors to the total portfolio
  • of attribution effects to the total value add
  • of time periods to the total attribution period

➢ As contrasted to geometric attribution methods…
Single Period Sector Performance...

Is easy. For Portfolio P:

<table>
<thead>
<tr>
<th>Period t</th>
<th>Return</th>
<th>Weight</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sector i</td>
<td>$R_{P,i,t}$</td>
<td>$W_{P,i,t}$</td>
<td>$C_{P,i,t} = W_{P,i,t} * R_{P,i,t}$</td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>$R_{P,t} = \sum_{i} C_{P,i,t}$</td>
</tr>
</tbody>
</table>
Multi-Period Sector Performance...

Is easy. For Portfolio P:

<table>
<thead>
<tr>
<th></th>
<th>Period 1</th>
<th>Period t</th>
<th>Full Performance Period 0 - t</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Adjusted Contribution</td>
<td>Adjusted Contribution</td>
<td>Adjusted Contribution</td>
</tr>
<tr>
<td>R W C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sector i</td>
<td>(\tilde{C}<em>{P,i,t} = C</em>{P,i,t} \times (1 + \tilde{R}_{P,i-1}))</td>
<td>(\tilde{C}<em>{P,i,t} = C</em>{P,i,t} \times (1 + \tilde{R}_{P,i-1}))</td>
<td>(\tilde{C}<em>{P,t} = \sum</em>{i} \tilde{C}_{P,i,t})</td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>(\tilde{C}<em>{P,t} = \sum</em>{i} \tilde{C}_{P,i,t})</td>
<td>(\tilde{C}<em>{P,t} = \sum</em>{i} \tilde{C}_{P,i,t})</td>
<td>(R_p = \sum_{i} \tilde{C}<em>{P,i} = \sum</em>{i} \tilde{C}_{P,t})</td>
</tr>
</tbody>
</table>
“Adjusted” contributions are scaled to prior cumulative Portfolio return:

\[
\bar{R}_{P,t} = \left[ \prod_{s=1}^{t} (1 + R_{P,s}) \right] - 1
\]

Consistent with intuition for dollar contributions, which are additive: 10% return on $100 = $10 in period 1 makes 10% return in period 2 “worth” $11, or 11% in base-period terms.
Single Period Sector Attribution...

Is easy.

<table>
<thead>
<tr>
<th>Period t</th>
<th>Portfolio P</th>
<th>Benchmark B</th>
<th>Attribution Effects</th>
<th>Value Added</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R W C</td>
<td></td>
<td>Allocation</td>
<td>Selection</td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td>$A_{i,t}$</td>
<td>$S_{i,t}$</td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>$A_i = \sum_i A_{i,t}$</td>
<td>$S_i = \sum_i S_{i,t}$</td>
</tr>
<tr>
<td></td>
<td>$V_{i,j} = C_{P,i,t} - C_{B,i,t}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= A_{i,t} + S_{i,t} + I_{i,t}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Single Period Attribution - 2

- Using the familiar, “vanilla” Brinson method:

\[ A_{i,t} = (W_{P,i,t} - W_{B,i,t}) \times R_{B,i,t} \]
\[ S_{i,t} = W_{B,i,t} \times (R_{P,i,t} - R_{B,i,t}) \]
\[ I_{i,t} = (W_{P,i,t} - W_{B,i,t}) \times (R_{P,i,t} - R_{B,i,t}) \]

- Many use Brinson-Fachler, in which:

\[ A_{i,t} = (W_{P,i,t} - W_{B,i,t}) \times (R_{B,i,t} - R_{B,t}) \]

- but then

\[ V_{i,t} = C_{P,i,t} - C_{B,i,t} \neq A_{i,t} + S_{i,t} + I_{i,t} \]
## Multi-Period Sector Attribution

Is hard!

<table>
<thead>
<tr>
<th></th>
<th>Period 1</th>
<th>Period t</th>
<th>Full Period Attribution 0 - t</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A S I V</td>
<td>A S I V</td>
<td>Allocation Selection Interaction Value Added</td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td>( \tilde{A}_i = ??? ) ( \tilde{S}<em>i = ??? ) ( \tilde{I}<em>i = ??? ) ( V_i = \tilde{C}</em>{P,i} - \tilde{C}</em>{B,i} ) ( = \tilde{A}_i + \tilde{S}_i + \tilde{I}_i )</td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sector i</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>( A = \sum_i \tilde{A}_i ) ( S = \sum_i \tilde{S}_i ) ( I = \sum_i \tilde{I}_i ) ( V = R_p - R_B ) ( = A + S + I )</td>
</tr>
</tbody>
</table>
It’s hard, because the standard Brinson formulas include weight & return from **two** entities, the Portfolio and the Benchmark.

What is the “adjustment” factor when these two entities do not track?
Solutions: A Simple Attempt

- Just use the prior cumulative Portfolio return, like we did with single period Portfolio performance:

\[
\begin{align*}
\tilde{A}_{i,t} &= A_{i,t} \times (1 + \bar{R}_{P,t-1}) \\
\tilde{S}_{i,t} &= S_{i,t} \times (1 + \bar{R}_{P,t-1}) \\
\tilde{I}_{i,t} &= I_{i,t} \times (1 + \bar{R}_{P,t-1})
\end{align*}
\]

- Not exact
- The further Portfolio and Benchmark returns drift, the worse it gets.
Something a Tad More Sophisticated?

- Scale the weights by their respective entity’s prior cumulative performance:

\[
\tilde{A}_{i,t} = [(W_{P,i,t} \times (1 + \bar{R}_{P,t-1})) - (W_{B,i,t} (1 + \bar{R}_{B,t-1}))] \times R_{B,i,t}
\]

\[
\tilde{S}_{i,t} = [(W_{B,i,t} (1 + \bar{R}_{B,t-1}))] \times (R_{P,i,t} - R_{B,i,t})
\]

\[
\tilde{I}_{i,t} = [(W_{P,i,t} \times (1 + \bar{R}_{P,t-1})) - (W_{B,i,t} (1 + \bar{R}_{B,t-1}))] \times (R_{P,i,t} - R_{B,i,t})
\]

- Still not exact

- There is an algebraic solution for the error, but it is hard to explain, and can be larger than the effect itself.

Attempts to solve by viewing continuously compounding effects

\[
\{\tilde{A}_{i,t}, \tilde{S}_{i,t}, \tilde{I}_{i,t}\} = \left[ \frac{\ln(1 + R_{p,t}) - \ln(1 + R_{B,t})}{R_{p,t} - R_{B,t}} \right] \times \{A_{i,t}, S_{i,t}, I_{i,t}\}
\]

But the approach still leaves an “unexplained residual … it is fair to distribute the residual proportionately”.

Hence, a final re-adjustment occurs after summing up the adjusted effects:

\[
\{\tilde{A}_i, \tilde{S}_i, \tilde{I}_i\} = \sum_t \left[ \{\tilde{A}_{i,t}, \tilde{S}_{i,t}, \tilde{I}_{i,t}\} \right] \left[ \frac{\ln(1 + R_p) - \ln(1 + R_B)}{R_p - R_B} \right]
\]
Menchero

- Based on geometric compounding, constructs a scaling factor, such that:

\[
\{\tilde{A}_{i,t}, \tilde{S}_{i,t}, \tilde{I}_{i,t}\} = F \{A_{i,t}, S_{i,t}, I_{i,t}\}
\]

\[
F = \frac{1}{T} \left[ \frac{R_p - R_B}{(1 + R_p)^{1/T} - (1 + R_B)^{1/T}} \right]
\]

- But again, “still leaves a small residual … introduce a set of corrective terms \(a_t\) that distribute this small residual among the different periods so that the following equation exactly holds”

\[
R_p - R_B = \sum_t (F * a_t) * (R_{p,t} - R_{B,t})
\]

- And proceeds by optimizing the residual to make \(a_t\) as small as possible


\[
\{\tilde{A}_{i,t}, \tilde{S}_{i,t}, \tilde{I}_{i,t}\} = \{A_{i,t}, S_{i,t}, I_{i,t}\} \times \left[ \prod_{j=1}^{t-1} 1 + R_{P,t} \right] + R_{B,t} \times \left[ \sum_{j=1}^{t-1} \{\tilde{A}_{i,t}, \tilde{S}_{i,t}, \tilde{I}_{i,t}\} \right]
\]
### Frongello, Wilshire - 2

<table>
<thead>
<tr>
<th>Sources of this period value added</th>
<th>This period portfolio return =</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>This period Benchmark</td>
</tr>
<tr>
<td></td>
<td>This period Allocation</td>
</tr>
<tr>
<td></td>
<td>This period Selection</td>
</tr>
<tr>
<td></td>
<td>This period Interaction</td>
</tr>
<tr>
<td>Cumulative Benchmark</td>
<td>Benchmark</td>
</tr>
<tr>
<td>Cumulative Allocation</td>
<td>Allocation</td>
</tr>
<tr>
<td>Cumulative Selection</td>
<td>Selection</td>
</tr>
<tr>
<td>Cumulative Interaction</td>
<td>Interaction</td>
</tr>
</tbody>
</table>

- Decomposes a periods attribution effect into:
  - This period’s effect * cumulative prior portfolio return
  - Plus cumulative prior periods’ effect * this period’s benchmark return
- Valtonnen later shows that this is a valid though arbitrary decomposition, and is one of a continuum of exact solutions
Davies & Laker

- Goes back to the “first principles” of Brinson, Hood, Beebower (1986), defining “Notional Portfolios”:

<table>
<thead>
<tr>
<th>Weights of Portfolio</th>
<th>Weights of Benchmark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Returns of Portfolio</td>
<td>Returns of Benchmark</td>
</tr>
<tr>
<td><strong>Portfolio</strong></td>
<td><strong>Notional Allocation</strong></td>
</tr>
<tr>
<td><strong>Notional Selection</strong></td>
<td><strong>Benchmark</strong></td>
</tr>
</tbody>
</table>

- In period $t$, then,

$$A_t = R_{A,t} - R_{B,t}$$
$$S_t = R_{S,t} - R_{B,t}$$
$$I_t = R_{P,t} + R_{B,t} - R_{A,t} - R_{S,t}$$
Compounded Notional Portfolios

- Davies & Laker called it the “Exact Brinson Method”
- Currently referred to by this more neutral moniker
- Stated that any linking methodology, however it works, should equal the results of CNP, or it isn’t Brinson
- Has intuitive appeal based on its real-world feasibility
But, as late as Summer of 2005, the primary downside of CNP was that no one had put forth a method of producing sector-level attribution effects that summed to the total portfolio effects.

- Actually, Laker himself showed an example using Cariño under CNP, but it wasn’t exact
- Valtonnen showed Frongello under CNP. Exact, but still a hybrid – and the interaction effect was a monster.
The Solution

- You’ve probably seen, however, that we already solved this problem back on page 4
- Since with CNP we are dealing with four \textit{individual} portfolios (even if two of them are notional), we can simply apply the multi-period single portfolio method to each of them, and apply the “first principles” Brinson:

\[
\begin{align*}
\widetilde{A}_{i,t} &= \widetilde{C}_{A,i,t} - \widetilde{C}_{B,i,t} \\
\widetilde{S}_{i,t} &= \widetilde{C}_{S,i,t} - \widetilde{C}_{B,i,t} \\
\widetilde{I}_{i,t} &= \widetilde{C}_{P,i,t} + \widetilde{C}_{B,i,t} - \widetilde{C}_{A,i,t} - \widetilde{C}_{S,i,t}
\end{align*}
\]

- And everything sums exactly every which way
CNP vs. Other Methods

- Robustness, Absence of Residuals:
  - Equivalent

- Intuitiveness:
  - Superior, IMHO

- Transparency:
  - Superior, by virtue of simplicity

- Commutativity:
  - “simply interchanging two of the periods should not change the results”.
  - Only Frongello is not commutative, and he argues that that is a desirable aspect, calling it “Order Dependence”
CNP vs. Other Methods - 2

- Metric Preservation
  - “Two periods that have identical relative performance should contribute equally to relative performance when they are linked together.”
  - This criteria, advanced by Menchero, is only evidenced in Menchero’s method

- A-causality
  - “August’s stock selection contribution to this year’s excess return does not become available until after the end of December”
  - Put another way, a report produced at the end of May will have different numbers for May’s attribution effects than a report produced at the end of June
  - IMHO, a big deal
  - Cariño and Menchero both exhibit a-causality
Biggest Remaining Issue with CNP:

- **Spurious Interaction Effects**
  - Interaction appears over multiple periods, even when no single period exhibits Interaction at the Total Portfolio level.
  - Laker later addresses persuasively, by pointing out that Interaction arises not only from simultaneous effect of Allocation and Selection, but also from combined effects over multiple periods.
  - Frongello has interesting example, where Interaction effects in separate sectors exactly cancel each other out. Can produce alarmingly large Interaction effects over multiple periods.