Robust Optimization:
What Works and What Does Not

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for

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Overview

• Why Optimized Portfolios Are Not Robust?
  - Ex-Post Performance and Multi-period Backtests

• Robust Optimization with SOCP (Second Order Cone Programming)
  - Equivalence to the Quadratic Penalty

  . Two-Stage Optimization
    - Make a Portfolio Optimization Process (POP) Robust

    . “Less is More”
      - One-period Static Optimization might be Over-Analyzed

• Closing Remarks
An Example

- Long-Only Active Portfolio Benchmarked to Russell 2000 Growth
- Max Alpha (Predicted)
- Tracking Error no more than 4.75%
- Security level active-weight bounds
- Beta neutral to the benchmark

- Tcost (25 bps impact + 3¢/share)
- No more than 200 securities in the portfolio
### Initial Portfolio

<table>
<thead>
<tr>
<th>Annualized Performance</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Active Returns</td>
<td>3.67%</td>
<td>3.71%</td>
<td>3.41%</td>
</tr>
<tr>
<td>Active Returns (after Tcost)</td>
<td>0.80%</td>
<td>0.84%</td>
<td>0.54%</td>
</tr>
<tr>
<td>Active Risk</td>
<td>4.62%</td>
<td>4.59%</td>
<td>4.74%</td>
</tr>
<tr>
<td>Tcost Impact</td>
<td>-2.89%</td>
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### Initial Portfolio

**Annualized Performance**

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<tr>
<td><strong>Active Returns</strong></td>
<td>0.50%</td>
<td>1.84%</td>
<td>0.06%</td>
</tr>
<tr>
<td><strong>Active Risk</strong></td>
<td>4.75%</td>
<td>4.81%</td>
<td>4.51%</td>
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**Tcost Optimization**

- **Initial A Rt=0.5%**
- **Initial B Rt=1.84%**
- **Initial Cash Rt=0.06%**
Does Robust Optimization help?


What is Robust MVO?

A mean-variance optimization (MVO) problem:

\[\begin{align*}
\alpha & \text{ is the expected absolute or excess return of stocks from a universe} \\
\max & \quad \alpha'w \\
\text{subject to:} & \quad w'Vw \leq \sigma^2 \\
& \quad e'w = 1 \\
& \quad w \geq 0.
\end{align*}\] (1)

Its active counterpart can be written as:

Here, \(\alpha\) is the expected active return of stocks from a universe.

\[\begin{align*}
\max & \quad \alpha'w \\
\text{subject to:} & \quad (w - w^b)'V(w - w^b) \leq \tau^2 \\
& \quad e'w = 1 \\
& \quad w \geq 0.
\end{align*}\] (2)

A very important and well-known property for the active problem is:

\[\alpha'w^b = 0.\]

A benchmark portfolio’s alpha is zero! (See Grinold and Kahn (1999)).
What is Robust MVO?

Due to the fact that the true $\alpha$ is not know, and we can only obtain an estimation of the “true” $\alpha$. Assume

$$\alpha \sim N(\alpha^*, \Omega)$$

a normal distribution of known covariance.

For any small probability $p$, we can construct a $(1 - p)$ confidence region for some $\kappa_p$ as

$$\mathcal{B} = \{\alpha|(\alpha - \hat{\alpha})'\Omega^{-1}(\alpha - \hat{\alpha}) \leq (\kappa_p)^2\},$$

where $\hat{\alpha}$ is the estimated expected return of the stocks. The robust optimization tries to optimize the “worst case scenario” out of all possibilities, or over some most likely cases.

$$\max_{\{\alpha \in \mathcal{B}\}} \min \quad \alpha'w$$

$$w'Vw \leq \tau^2$$

$$e'w = 1$$

$$w \geq 0.$$  \hspace{1cm} (3)
What is Robust MVO?

**Absolute Return-Risk Robust MVO Formulation**

The robust optimization problem can be formulated as,

\[
\max_w \quad \hat{\alpha}w - \kappa_p \sqrt{w'\Omega w} \\
\text{subject to} \quad w'\Sigma w \leq \sigma^2 \\
e'w = 1 \\
w \geq 0.
\]

This is a second-order cone programming (SOCP) problem instead of a quadratic programming problem (QP) largely due to the appearance of the square root.

Because

\[
\begin{align*}
\min_{\alpha} \{ \alpha'w \mid (\alpha - \hat{\alpha})'\Omega^{-1}(\alpha - \hat{\alpha}) \leq (\kappa_p)^2 \} \\
= \hat{\alpha}w + \kappa_p \min_x \{ x'u \mid x'x \leq 1 \} \\
= \hat{\alpha}w - \kappa_p \|u\| \\
= \hat{\alpha}w - \kappa_p \sqrt{w'\Omega w}.
\end{align*}
\]

\[
\max_{\{\alpha \in B\}} \min_{\alpha} \quad \alpha'w = \max \quad \hat{\alpha}w - \kappa_p \sqrt{w'\Omega w}.
\]
Our Results

Demystification of the Robust Optimization

$$\max \ \hat{\alpha} w - \kappa \sqrt{w' \Omega w}$$
$$w' V w \leq \sigma^2$$
$$e' w = 1$$
$$w \geq 0.$$  \hspace{2cm} (6)

There exists a $\lambda$ such that the solution of the following problem has an identical solution to the above Robust MVO,

$$L(\lambda) = \max \ \hat{\alpha} w - \lambda w' \Omega w$$
$$w' V w \leq \sigma^2$$
$$e' w = 1$$
$$w \geq 0.$$  \hspace{2cm} (7)

This is a Quadratic Penalty Function problem.
Our Results

Why Our Claim is true?

\[
\begin{align*}
\max \quad & \hat{\alpha}'w \\
& w'Vw \leq \sigma^2 \\
& e'w = 1 \\
& w \geq 0.
\end{align*}
\]

It is well-known that the regular MVO can be solved by

\[
\begin{align*}
\max \quad & \hat{\alpha}'w - \mu w'Vw \\
& e'w = 1 \\
& w \geq 0,
\end{align*}
\]

for some \( \mu \). (\( \mu \) is formally called the Lagrangian Multiplier.)

\[
\begin{align*}
\max \quad & \hat{\alpha}w - \kappa \sqrt{w'\Omega w} \\
& w'Vw \leq \sigma^2 \\
& e'w = 1 \\
& w \geq 0.
\end{align*}
\]

By the same logic, the above Robust Problem can be solved through

\[
\begin{align*}
\max \quad & \sqrt{w'\Omega w} \\
& \sqrt{w'\Omega w} \leq \theta \quad \text{or} \quad w'\Omega w \leq \theta^2 \\
& w'Vw \leq \sigma^2 \\
& e'w = 1 \\
& w \geq 0.
\end{align*}
\]
Our Results

The Implication of Our Claim

Since $V$ is the risk model, $\alpha^*$ is the expected absolute returns,

$$\alpha \sim N(\alpha^*, V).$$

On account of Axioma’s Robust model’s assumption $\alpha \sim N(\alpha^*, \Omega)$, we shall conclude that

$$\Omega = \delta V.$$  

If this is the case, the Efficient Frontier from the Robust Optimization would be identical to the one from Regular MVO.

\[
L(\lambda) = \max_{w'Vw \leq \sigma^2, \ e'w = 1, \ w \geq 0} \tilde{\alpha}w - \lambda \delta w'Vw
\]  

(8)
A Big Question

Does Robust MVO Enhance Portfolio Alpha?

Just consider a simpler case where $\Omega = diag(\sigma^2_i)$.

$$\begin{align*}
\max & \quad \tilde{\alpha} w - \lambda \{ w' \Omega w \} \\
& \quad w' V w \leq \sigma^2 \\
& \quad e' w = 1 \\
& \quad w \geq 0.
\end{align*}$$

$$\begin{align*}
\max & \quad \tilde{\alpha} w - \mu \quad w' (V + (\lambda/\mu) \Omega) w \\
& \quad e' w = 1 \\
& \quad w \geq 0.
\end{align*}$$

The Robust Optimization suggests use a different risk model that increases the idiosyncratic risks for stocks of which the alphas might be harder to predict.
A Mistake in the Current Robust MVO Formulation

Active Returns/Risks Robust Reformulation

For active returns MVO, the corresponding robust problem is

$$\begin{align*}
\max_{\{\alpha \in B\}} & \quad \alpha' w \\
(w - w^b)' V (w - w^b) & \leq \tau^2 \\
e' w & = 1 \\
w & \geq 0.
\end{align*}$$

(11)

Ceria and Stubbs' (2005) from Axioma claimed that the above objective function is equivalent to

$$\max \ \tilde{\alpha} (w - w^b) - \kappa_p \sqrt{(w - w^b)' \Omega (w - w^b)}.$$

Actually, due to the fact that $\alpha' w^b = 0$, their objective is not entirely accurate. We show that a correct form of objective function should be
A Mistake in the Current Robust MVO Formulation

\[
\max \quad \hat{\alpha}w - \kappa_p \sqrt{w' \Omega w - \frac{(w' \Omega w^b)^2}{w^b' \Omega w^b}},
\]

or

\[
\max \quad \hat{\alpha}w - \lambda(w' \Omega w - \frac{(w' \Omega w^b)^2}{w^b' \Omega w^b})
\]
\[
(w - w^b)' \Sigma (w - w^b) \leq \sigma^2
\]
\[
e'w = 1
\]
\[
w \geq 0.
\]

(12)
What is the Remedy?

• Robust MVO does not differ much from regular MVO, it is equivalent to the quadratic penalty function method;

• Robust MVO requires to estimate another set of parameters;

• Robust Optimization is a one-period static solution.
Two-Stage Optimization --- A Solution

- **Stage 1**: Find a path-independent "ideal" portfolio
  - Max Alpha
    1a. The Tracking Error upper bound;
    1b. Linear side-constraints; (Such as security-level bounds, factor-bets);
    1c. Upper bound on number of securities (optional).
- **Stage 2**: Find the *tradable portfolio*
  Max Utility Function
  
  same constraints as Stage 1, 1c) shall be included.
  
  Utility Function = $\alpha - \lambda \text{ (tracking_err vs. } \text{ideal portfolio})^2$
  
  - $\mu$ (Tcost against the *legacy portfolio*)
Two-Stage vs. One-Stage

Two-Stage Opt vs One-Stage Tcost Opt

Cumulative Active Return vs. R20000G

Initial A 2-Stage Opt Rt=1.82%
Initial B 2-Stage Opt Rt=1.66%
Initial Cash 2-Stage Opt Rt=1.94%
Initial A Tcost Opt Rt=0.50%
Initial B Tcost Opt Rt=1.84%
Initial Cash Tcost Optf Rt=0.06%
Two-Stage Optimization
Two-Stage Optimization --- A Solution

- **tradable portfolio**

  \[ = \text{Function ( alpha, risk model, bounds, legacy portfolio, } \lambda, \mu) \]

- How to decide \( \lambda \) and \( \mu \) is an art.
Two-Stage Optimization

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Two-Stage Optimization

Active Returns 1.84% 1.66% 1.94%
Active Risk 4.75% 4.81% 4.51%
Tcost Impact -0.20% -0.21% -0.26%

One-Stage Tcost Optimization

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Two-Stage vs. One-Stage
Two-Stage vs. No-Tcost Optimization
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Less is More

• It might not be a good idea to over-analyze the one-period static optimization problems

• How to make a Portfolio Optimization Process (POP) Robust?
  1) form a maximization problem that is concave if possible;
     having a unique optimal solution is even better (When the risk model is a positive-definite matrix and risk budget is binding).

  2) Constraints that limit number of securities in a portfolio may cause the optimization problem non-concave, risk-targeting is a bad idea.

  3) Find and understand a series of path-independent portfolios. The PMs don’t need to trade on these portfolios, but they are better to know these portfolios.

  4) Including transaction cost and market impact into the POP, or adding turnover constraints may create a series of path-dependent portfolios.
• The Portfolio Optimization Process (POP) uses ex-ante data to achieve an ex-post goal, and it is a single-period proxy to a multi-period stochastic problems.

• The MVO pioneered by Markowitz was developed originally to trade-off between risks and returns.

• Currently Optimizer is used as a portfolio construction tool, sometimes, the tool to create a *tradable portfolio*. 
Recommendations

Run a Two-Stage Optimization (It requires an optimizer that can handles the 2nd Benchmark.)

If not, run an optimization ignoring Tcost/Turnover to ensure path-independent.

Closing Remarks

- Robust MVO is equivalent to the Quadratic Penalty Function Approach
- “Less is More”
- Two-Stage Optimization Enhances Robustness