Intra-Horizon Risk

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Overview

- Criticisms of tracking error / VaR
  - Tail shape
  - Time dimension
- Introduction of first-passage probability
  - Derivation in Normal, no drift case
  - Sidetrack 1: time-diversification argument
- Possible Extensions
  - Drift, jumps, stochastic volatility
- Introduce Jump Models
  - Review three common jump models, empirical evidence
  - Sidetrack 2: predictability of jumps
  - Solving first-passage probability for jump models
  - Empirical VaR intra-horizon versus end-of-horizon for three jump models
- Conclusion:
  - Implementable intra-horizon VaR / t.e. measure under assumptions
  - Roadmap for intra-horizon risk under more flexible distributions

Criticisms of VaR/tracking error

- Tail Shape: says nothing about size of loss, only probability of loss
- Time Dimension: says nothing about losses prior to horizon, recovery etc
- Distributional assumptions: doesn't allow for jumps, non-Normal behavior
- Time-scaling issues

Tail shape

- VaR result is independent of tail shape
- For measures that specifically include tail risk see:
  - Tail-conditional expectation, worst conditional expectation
    - Artzner, Delbaen, Eber and Heath (1997, 1999)
  - Expected shortfall, CVaR etc
    - Acerbi, Tasche (2001)

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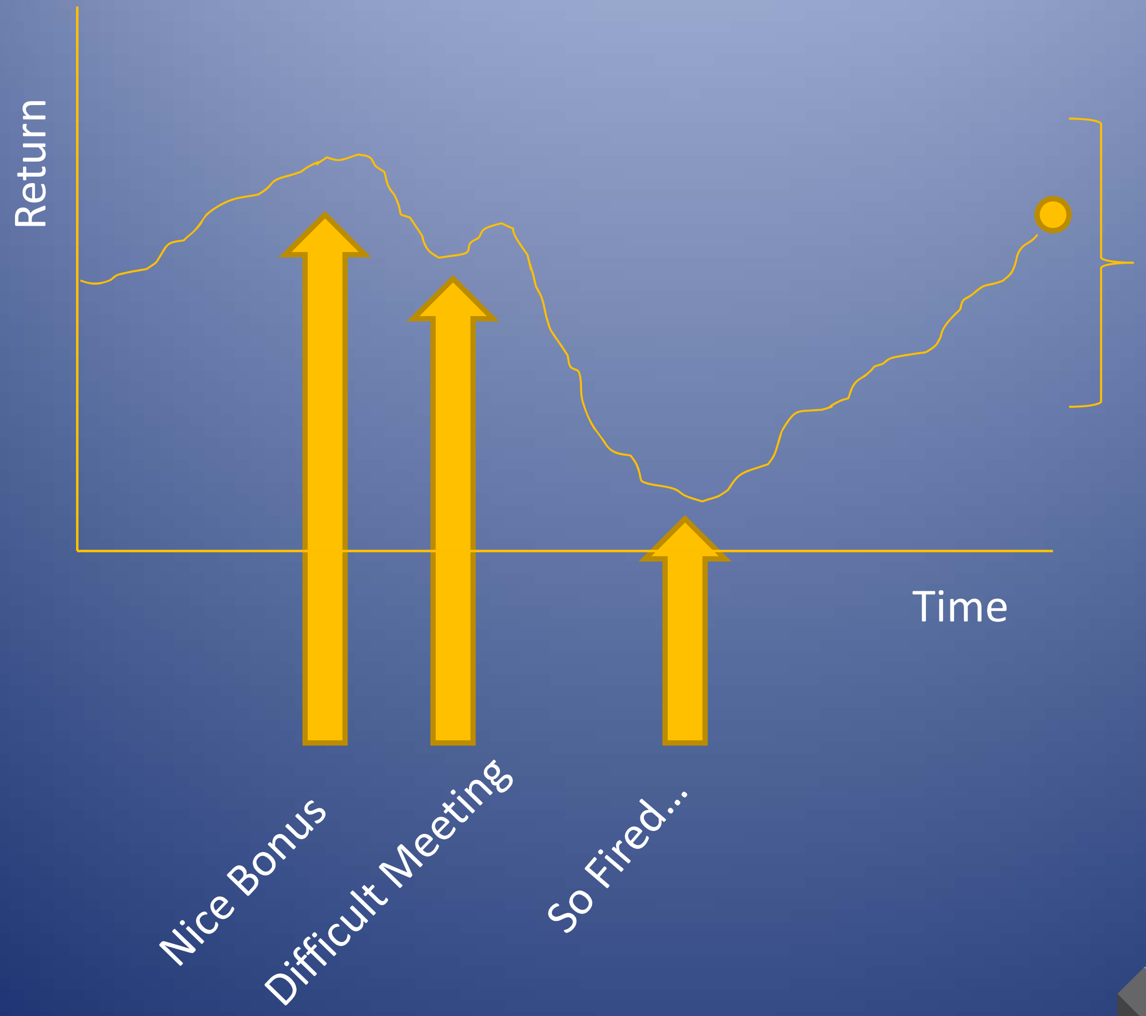
Time Dimension

- VaR/tracking error says nothing about the profit and loss distribution over the trading horizon
  - Basel Committee (1996)
  - Denote “intra-horizon risk”
  - Largely overlooked topic, few papers:
    - Stultz (1996)
    - Kritzman and Rich (2002)
    - Boudoukh, Richardson, Stanton, Whitelaw (2004)
  - Restrictive Assumptions: Brownian motion, no jumps
    - Note: Basel Committee (1996) also recognizes jumps are relevant for risk management purposes.
    - Bakshi, Panayatov (forthcoming JFE)

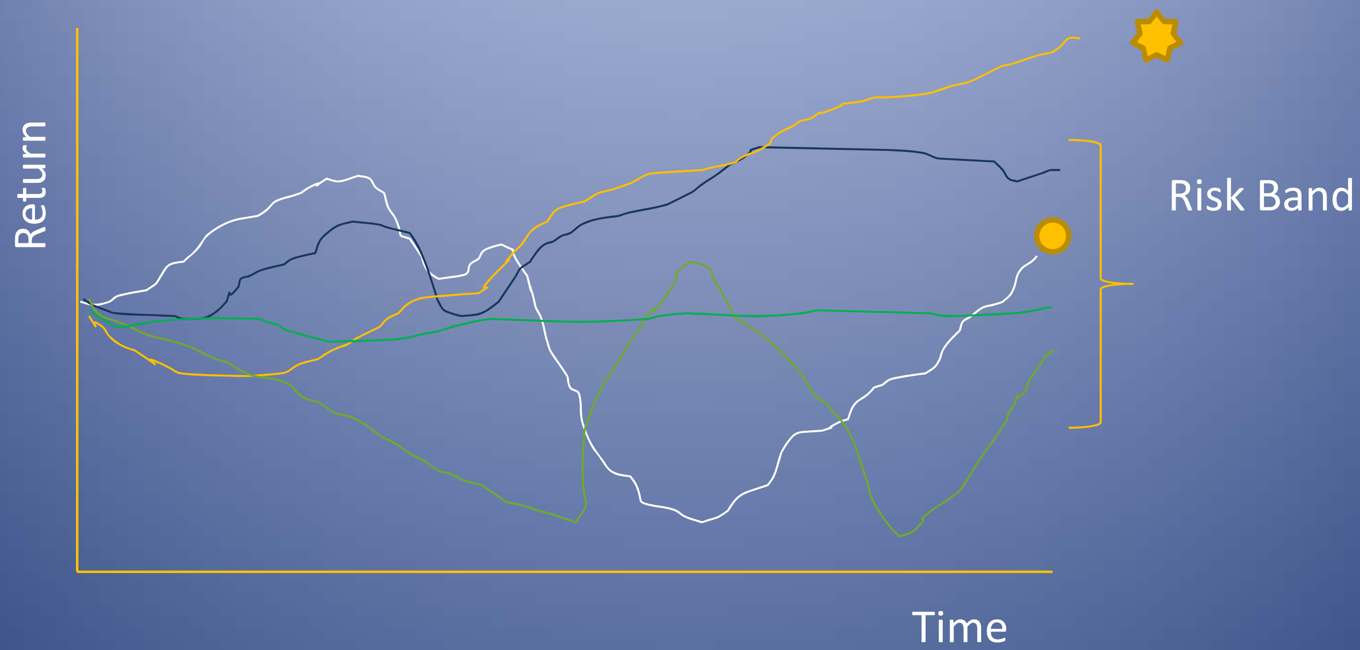
Motivation

Tracking error or VaR are concerned about the likely distribution of returns at the end of some investment period. In this example we have scored a goal – the final return is within the risk bands we set ourselves.

Motivation II



Another View



In this example there are five return paths. Only one path ends up outside our risk bands (and that on the "right" side) so by EOH measures the probability of breaching the band is 1/5. If we look throughout the horizon, 3 out of 5 paths breach the barrier at some point during the horizon – and hence the intra-horizon probability of breach is 3/5

Tracking Error, VaR

- End of Horizon Measure
- Characterizes the distribution of returns at the *end* of an investment period
- Says nothing about:
  - Maximum loss at some probability during the period
  - Probability of recovery following a loss during the period
  - …or anything else about the return path

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Risk horizon should match investment horizon

- "It should be obvious, but probably isn't, that our forecast horizon for our risk model should match our forecast horizon for our alphas, and implicitly therefore our holding period"
  – Rosenberg & Guy (1975)

- What Rosenberg neglects is the very human characteristic of second-guessing the process mid-period & the idea that we might be leveraged and therefore unable to put our feet on the wheel and wait for it all to turn out right in the end

Who Cares?

- **Survival Analysis** – Leveraged Investor: if there is a floor below which I cannot go, I should care greatly what the probability of hitting that floor is during the period
- **Monitored Investor** – if there is a good chance I will lose a client if performance is worse than a particular hurdle at any time during the year, I should care about the probability of hitting that hurdle

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Who Cares II

- Loan agreement – collateral
- Securities lending – collateral
- Capital adequacy
- Retirement/Endowment – funding a liability intra-horizon

Focus for Today

- Looking at the time-dimension
- What happens to the value of a position/portfolio *during* an investment horizon
- Compare results to standard end-of-horizon measure
  - We are using end-of-horizon VaR calculated using the Normal distribution, zero drift, and no jumps as our reference point. All multipliers are with respect to that measure.

Not a new issue

- We can think about the issue in two ways:
  - The size of a loss given a probability (i.e. VaR)
  - Turn that on its head: the probability of hitting a given loss level (barrier)
- Barrier Options since 1960's
- Merton, Reiner, Rubenstein (1973)
  - analytical closed form pricing for barrier options
  - (same problem: need to figure out probability of hitting a barrier during an investment horizon)
- Hirsa and Madan (2003)
- Kudryavtsev and Levendorskii (2006)

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Concept – First-Passage

- By using the concept from statistics of first-passage time (also known as first-hitting time) we can estimate the probability that our return will hit a particular hurdle e.g. -10% during the investment period, or estimate the maximum loss –$X during the period at some given confidence level e.g. 99%

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Caveats

- This does nothing to fix the issue of “strategy risk” – drift in the mean - to be discussed separately today
- This measure is equally exposed to estimation error as with the “regular” tracking error / VaR measure
- [*and equally benefits just the same from measures to address estimation error*]
- We are saying nothing about asymmetry of preferences
- We are staying away from saying anything about benchmarks being suitable or relative versus absolute risk
- To first get an intuitive tractable solution we assume Normality and zero drift, although we later relax those assumptions

Conclusions First

(After this slide you can check your Blackberry and not miss anything important)

- Intra-Horizon VaR (VaR-I) is larger than End-of-Horizon VaR
- Normal, no jumps: VaR-I = 1.107 * VaR[1]
- Non-Normal, allowing jumps: that multiplier can be up to 2.64[2]
- Non-Normal, jumps, stochastic vol, drift… it can get worse…

[1] Feller (1971)

[2] Bakshi and Panayatov (forthcoming, JFE)

Probability of loss at EOH

$$\Pr E = N\left(\left(\ln(1+L) - \mu T\right) \Big/ \sigma\sqrt{T} \right)$$

- Assumptions
  - Normal
  - Stationary vol
  - i.i.d.
  - (so we can use sqrt T)
- This is just the difference between the cumulative % loss and the cumulative expected return divided by the cumulative standard deviation, then apply N() to convert this standardized distance from the mean into a probability estimate.
- (Note I am cheating already and including a drift term)

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VaR

$$\Pr E = N\left({\left(\ln(1+L) - \mu T \right)} \Big/ {\sigma\sqrt{T}} \right)$$

- By rearranging the previous equation and solving for L (the loss) we get:

$$VaR = -\left(e^{\left(\mu T - Z\sigma\sqrt{T} \right)} - 1 \right) W$$

Probability of Loss “Intra Horizon”

$$\Pr I = \Pr E + N\left(\left(\ln(1+L) + \mu T\right) \Big/ \sigma\sqrt{T} \right)(1+L)^{2\mu / \sigma^2}$$

- Second part never zero or negative
  - Implies IH loss estimate > EOH estimate (always)
  - IH P(loss) rises as investment horizon expands, whereas EOH P(loss) declines as investment horizon expands
    - => time-diversification argument

[Side Track 1]: Time Diversification

- Samuelson (1963) argued against time diversification:
  - Although EOH p(loss) declines as inv hor incr., that benefit is offset by increasing magnitude of potential loss
  - Others have argued that time diversifies risk because the result does not depend on the magnitude of the potential loss
- WH risk presents another facet to that argument:
  - Risk increases even if investors only care about probability of loss

Solving the Problem

- First show path-independent (i.e. end of horizon) probability of loss
- Then use principle of reflection to convert path-dependent (i.e. within horizon) probability of loss into an equivalent path-independent (i.e. end of horizon) measure
- Review implications and assumptions

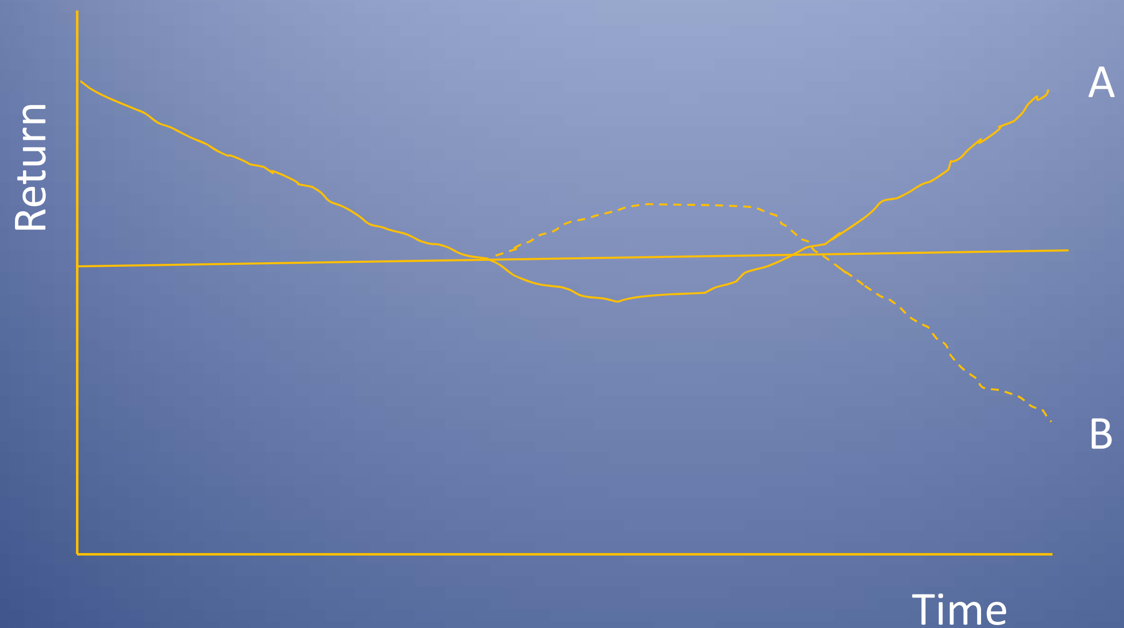
Path-independent probability of return r after time T

- Assume continuous returns are normally distributed:
  - Probability that cumulative return = r

$$P(r) = N\left[\frac{\left(r/n\right)}{\sigma}\right]$$

  - Where:
    - n is the number of years in T
    - r is the continuous cumulative return
    - σ is the standard deviation of the continuous returns

Principal of Reflection



Invoking the principal of reflection; the path-*dependent* probability of breaching a barrier and subsequently recovering to “A” [solid line] is the same as the path-dependent probability of breaching the barrier and then descending to B, which is the same as the path-*independent* probability of ending up at B, because B cannot be reached without breaching the barrier at some point.

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The Main Point

- Joint Prob (breach barrier b, and recover to A) = prob (end at B)
- The reflection principle allows us to convert a path-dependent probability into a path-independent probability under some assumptions

Set up the probabilities

- Prob (breach barrier)
  - = 1 – prob(never breach)
  - = 1 – joint prob(end up above, and min always above)
  - = 1 – prob(end above) + joint prob(end above, and min <=barrier)
  - = prob(end below) + joint prob(end above, and min <=barrier)
- By principal of reflection: the last term is equal to the path independent probability of reaching a return r below the barrier

$$\Pr(breach\_barrier) = N\left[\frac{\left(b/n\right)}{\sigma}\right] + N\left[\frac{\left(r/n\right)}{\sigma}\right]$$

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Assumptions so far

- Zero drift
- Normal distribution / Brownian motion
- Stationary volatility
- No jumps

Implications so far

- Within-horizon risk increases with time, adding fuel to the debate against time-diversification, because second term always positive
- Even assuming Normal distribution, stationary volatility, no jumps, and no drift, the 1% VaR-I is 10.7% bigger than 1% VaR
  - Feller (1971)
- At 2.5% it's 14.4% bigger, and at 5% it's 19.2% bigger
  - Boudoukh, Richardson, Stanton, Whitelaw (2004)

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Relaxing the zero drift assumption

- Girsanov's theorem
  - Useful transformation: using Girsanovs theorem we can transform a Brownian motion with drift into a Brownian motion without drift to make the maths easier.
  - The first-passage probability for a Brownian motion with drift is well known. See for example Karatzas and Shreve (1991)

$$\Pr(breach\_barrier) = N\left[\frac{\ln\left(B/S\right) - \mu T}{\sigma\sqrt{T}}\right] + \left(\frac{B}{S}\right)^{2\mu/\sigma^2} N\left[\frac{\ln\left(B/S\right) + \mu T}{\sigma\sqrt{T}}\right]$$

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The Impact of Drift

- For comparative purposes, look at what happens when we introduce drift of 10% and 15% respectively.

PROB	μ	VaR	VaR-I	VaR-I/VaR
5%	10%	$1.053\sigma\sqrt{T}$	$1.493\sigma\sqrt{T}$	1.417
2.5%	10%	$1.368\sigma\sqrt{T}$	$1.752\sigma\sqrt{T}$	1.281
1%	10%	$1.735\sigma\sqrt{T}$	$2.067\sigma\sqrt{T}$	1.191
5%	15%	$0.720\sigma\sqrt{T}$	$1.262\sigma\sqrt{T}$	1.753
2.5%	15%	$1.035\sigma\sqrt{T}$	$1.504\sigma\sqrt{T}$	1.453
1%	15%	$1.401\sigma\sqrt{T}$	$1.801\sigma\sqrt{T}$	1.285

Boudoukh, Richardson, Stanton, Whitelaw (2004)

What about jumps?

- Basel 1996: recognizes jumps are important
- other evidence:
  - Evidence for jumps:
    - Bakshi, Cao, Chen (1997)
    - Bates (2000)
    - Anderson, Benzoni, Lund (2002)
    - …but difficulty explaining market crashes with return jumps and diffusive volatility…
  - Evidence for large jump risk premia
    - Pan (2002)
  - Evidence for jumps in return *and* in volatility
    - Eraker, Johannes, Polson (2003)
  - For two examples of models with return and volatility jumps see
    - Duffie, Pan, Singleton (2000)

Jump Models

- Jump processes:
  - Merton (1976)
  - Carr, Geman, Madan, Yor (2002)
  - Carr and Wu (2003)
- Two groups:
  - finite jumps (i.e. infrequent)
  - infinite jumps (i.e. lots)

Sidetrack 2: predictability of jumps

- Strong evidence for predictability of jumps:
  - (For US markets) the VIX can predict both jump arrival and size
    - Johannes, Kumar, Polson (1999)
- Strong evidence of jump clustering
  - Implies that Merton (1976) model is not useful as it cannot capture persistence in the data i.e. jump clustering

The Approach

- Some analytical characterizations are known, particularly for stable processes with one-sided jumps:
  - Kyprianou (2006)
- In general, need to resort to PIDE approach [partial integro-differential equations]
  - Solve using finite difference methods
  - Or Monte Carlo
    - Atiya & Metwally (2002) – very fast approach, around 100 times faster than pure MC by leveraging a “Brownian bridge” to reduce the number of points that need to be calculated.
- Beyond this point I am out of my depth and will be visible only by bubbles on the surface…

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Some results

	Average VaR Multiples				Maximum VaR Multiples				Average VaR-I Multiples				Maximum VaR-I Multiples		
	JD	CGMY	FMLS		JD	CGMY	FMLS		JD	CGMY	FMLS		JD	CGMY	FMLS
S&P 500	1.21	1.33	1.44		1.33	1.37	1.69		1.60	1.39	1.94		1.77	1.50	2.08
FTSE	1.20	1.21	1.35		1.38	1.44	1.50		1.50	1.38	1.85		1.70	1.55	2.07
Nikkei	1.14	1.11	1.39		1.19	1.25	1.75		1.28	1.27	1.80		1.40	1.34	2.12
ATM Call	1.37	1.29	1.68		1.39	1.31	1.85		1.61	1.45	2.55		1.67	1.47	2.64

Compared to standard Normal VaR

JD = Merton's jump-diffusion model

CGMY is the two-sided pure-jump Levy model of Carr, Geman, Madan, and Yor

FMLS is the finite-moment log-stable model of Carr and Wu

(Bakshi and Panayatov)

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Implications

- VaR with jumps is bigger than N() VaR
  - We suspected as much beforehand…
- The choice of model makes a difference
  - But do they all fit the data equally well?
- VaR-I is consistently greater than VaR
  - We "knew" that beforehand, based on the known result with the Normal, no drift case
- VaR-I can be more than *double* standard VaR
  - And this goes some way toward justifying the Basel multipliers

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Stochastic Volatility and Jumps

- Just a taste…
  - Stochastic volatility in a VaR context:
    - Eberlein, Kallsen, and Kristen (2002)
  - Two-dimensional PIDEs for models with both jumps and stochastic volatility:
    - Feng and Linetsky (2006)
- Can be applied to first-passage calculations.
- (…let me know how you get on…)

Related Measures

- Leaning heavily once again on the barrier-option analogy we could also consider:
  - A risk *range* rather than a single loss level (double barrier)
  - A *time-varying* barrier or range
  - A barrier that varies as a function of volatility
- E.g. Lo & Hui (2007), pricing double barriers where volatility, dividend yield, and the barriers are stochastic.

Conclusions

- Intra-Horizon Risk is important, and neglected
- If you are a buy-side portfolio manager you can probably implement and use N() no drift *today*.
- If you are a levered investor or a trading desk, follow the yellow brick road of references laid out today, put some work into it, and be safe.

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