Simple and Robust Risk Budgeting Using TEES

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Agenda

Risk Budgeting is Portfolio Optimization by Another Name

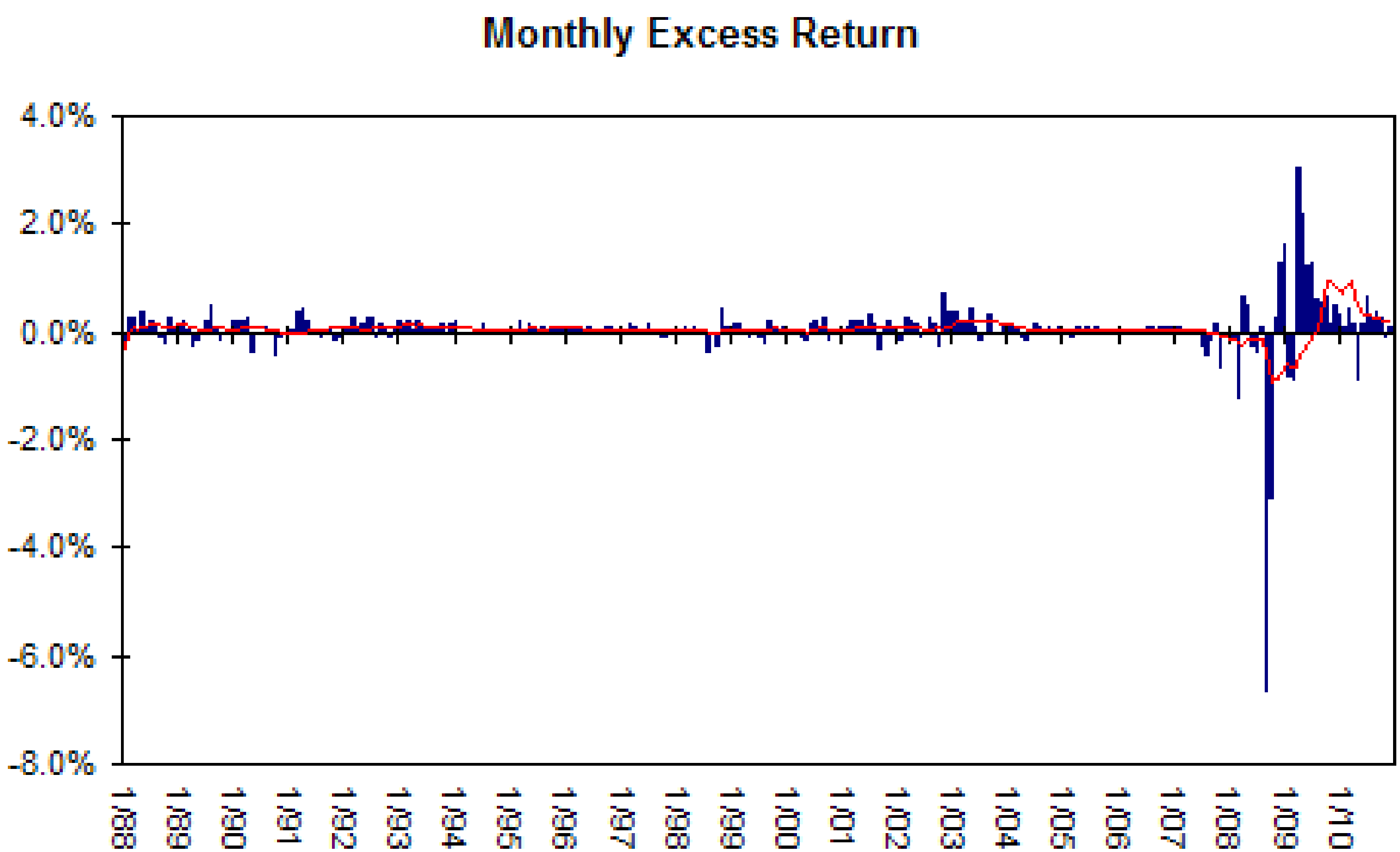
- I got interested in risk budgeting when it entered our investment process after a merger
  - Risk Budget = vol contribution of a strategy we want to include in our portfolio = $x_i \times \sigma_i$

- Basic problem setup
  - CIO / product head has high level views on asset classes, alpha teams and strategies
  - Views on individual assets typically reside with a specialist

- We need to solve a two level allocation / optimization problem
  - CIO / Product head optimally allocates risk to alpha teams and strategies
  - Alpha teams allocate active risk between individual securities and sub-strategies

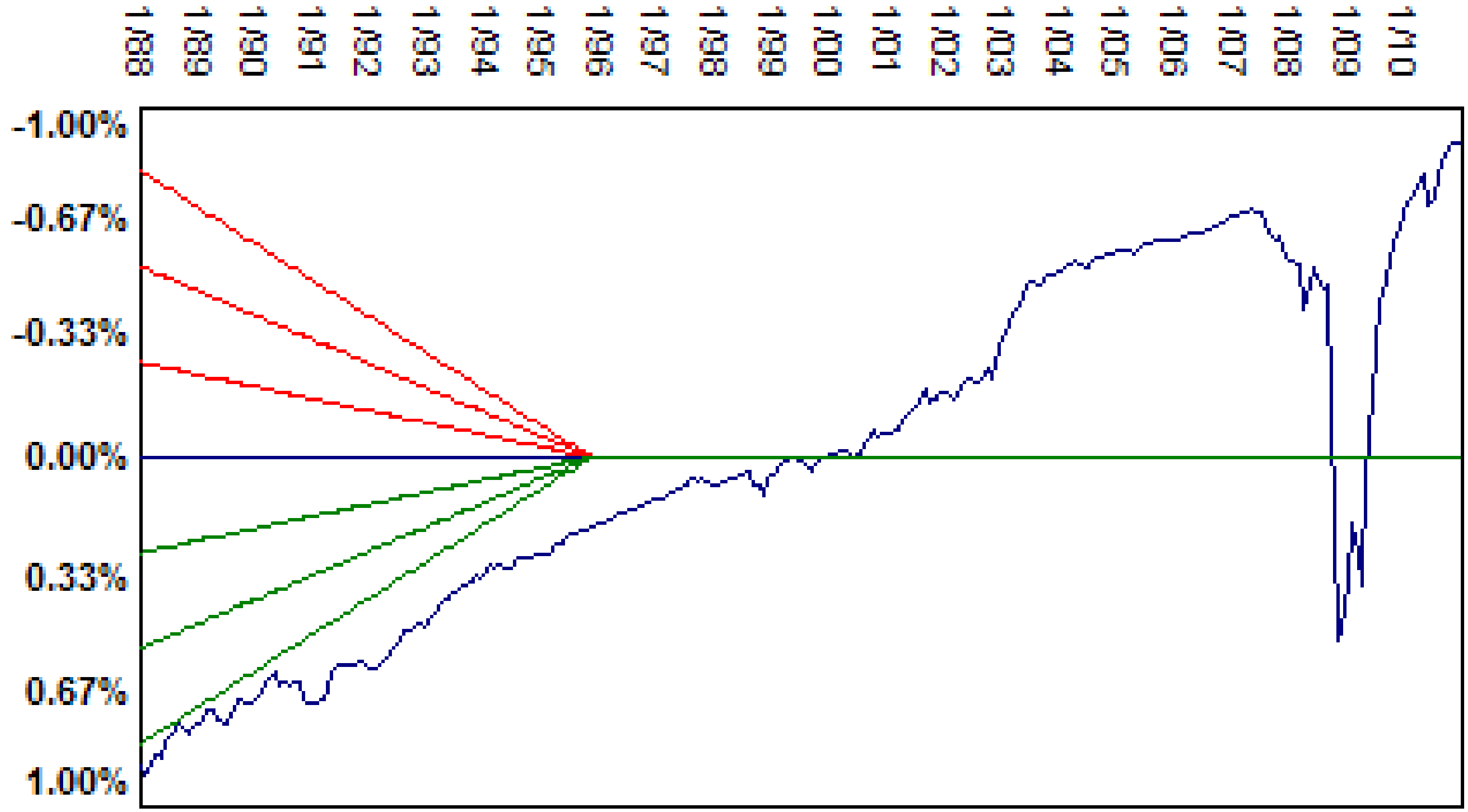
- In this context, risk budgeting feels more natural than portfolio optimization
  - But risk budgeting is most often done in a mean-variance framework
  - And fixed income is loaded with tail risk, both explicit (options) and implicit (credit, liquidity)

Tail Risk: Short Term Corporates vs. Treasuries



Source: Merrill Lynch MLX Global Index System and BNP Paribas Investment Partners

Cum. Excess Return: Short Term Corporates vs. Treasuries



Source: BNP Paribas Investment Partners

Risk Budgeting – Needs, Wants and What's Available

- I needed to create risk budgets in a user friendly and computationally tractable way
  - Account for tail risk in an intuitive way without requiring a huge infrastructure
  - OK to work at strategy level, no need to drill down to individual securities

- None of the traditional enhanced Mean-Variance approaches met my needs
  - Resampling (Michaud and Michaud)
  - Shrinkage (Jobson and Korkie, Ledoit and Wolf)
  - Direct estimation of the utility function via simulation (Sharpe)
  - Robust optimization (Ceria)
  - Black-Litterman (Black and Litterman)

- ETL Optimization (Rachev and Martin) met many of my needs but was too complex

- My requirements practically mandated a different approach
  - First need to define exactly how we want to think about risk

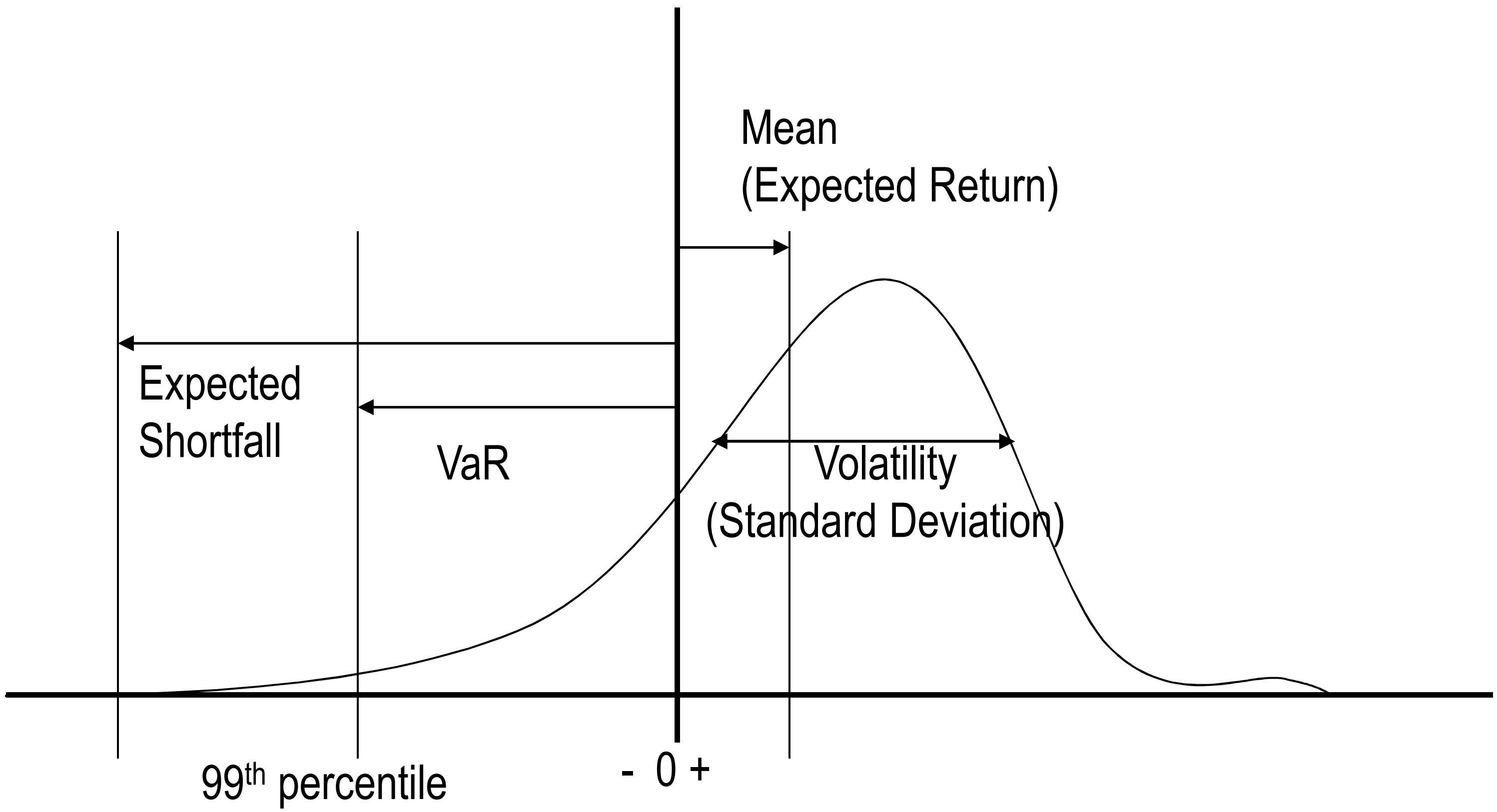
Redefining Risk: Artzner, Delbaen, Eber, Heath, 1997

- Risk is synonymous with loss – it's not the same as uncertainty in return
  - Risk is a measure of the amount of cash needed to support a position / a strategy / a portfolio

- A good risk measure for an asset / a strategy / a portfolio X satisfies these axioms
  - Relevance: If Return (X) is sometimes < 0 , then Risk (X) > 0
  - Monotonicity: If Return (X) is almost surely <= Return (Y), then Risk (X) >= Risk (Y)
  - Positive Homogeneity: Risk (c*X) = c * Risk (X)
  - Translation Invariance: Risk (X + cash) = Risk (X) – cash
  - Subadditivity: Risk (X + Y) <= Risk (X) + Risk (Y)

- A risk measure that satisfies these criteria is called a *coherent* measure of risk
  - Standard deviation is not coherent (Does not satisfy monotonicity)
  - VaR is not coherent (Does not satisfy subadditivity)
  - Expected Shortfall is coherent (Satisfies all the axioms)

A Distribution and its Risk Measures



Expected Shortfall = Expected loss conditioned on return < VaR

We want to use a coherent measure of risk to guide our risk budgets

Review: MV Risk Budgeting With Independent Alpha Sources

- Blitz and Hottinga: assume we have a beta portfolio and *N* alpha portfolios
  - The alpha portfolios are independent of the beta portfolio and of each other
  - The i^{th} alpha portfolio has an information ratio of IR_i

- How should we budget our risk to achieve a tracking error of TE_{Target}?

- Blitz and Hottinga solve Markowitz's mean variance equations to get

$$Risk\ Budget_i = \frac{IR_i}{\sqrt{IR_1^2 + IR_2^2 + \cdots + IR_N^2}} \times TE_{\mathrm{Target}}$$

- Risk budgets depend only on *relative* Information Ratios and Target *TE*!

- Important special case: If all *IR's* are equal, all risk budgets are equal to $\dfrac{TE_{\mathrm{Target}}}{\sqrt{N}}$

Risk Budgeting With Independent Alpha Sources: Example

- Simple example: 2 alpha sources, each with an *IR* of 1, TE_{Target}= 100 bp
  - Risk budget for alpha source 1 = $\frac{1}{\sqrt{1^2+1^2}}\times 100$ = 71 bp
  - Risk budget for alpha source 2 = $\frac{1}{\sqrt{1^2+1^2}}\times 100$ = 71 bp
- What if the information ratios were 0.5 and not 1?
  - Risk budget for alpha source 1 = $\frac{0.5}{\sqrt{0.5^2+0.5^2}}\times 100$ = 71 bp
  - Risk budget for alpha source 2 = $\frac{0.5}{\sqrt{0.5^2+0.5^2}}\times 100$ = 71 bp
- Change the information ratios to 0.5 and 1. What happens now?
  - Risk budget for alpha source 1 = $\frac{0.5}{\sqrt{0.5^2+1^2}}\times 100$ = 45 bp
  - Risk budget for alpha source 2 = $\frac{1}{\sqrt{0.5^2+1^2}}\times 100$ = 89 bp

Risk Budgeting With Independent Alpha Sources: Thoughts

- The Good
  - Incredibly simple and practical!!! Can create a risk budget on the back of an envelope!

- The Bad
  - Does not account for tail risk, completely ignores correlations

- The Challenge: Account for tail risk in a way that's intuitive and usable
  - Allocating Expected Shortfall between strategies is VERY hard
    - Typically need to make lots of assumptions, and even so, requires lots of computations
    - And clients typically have guidelines on tracking error, not Expected Shortfall

- The Solution: Keep one foot in the old world, the other in the new
  - Start with a MV optimal risk budget, then modify it using information on Expected Shortfall
  - Similar in spirit to Black-Litterman, but less formal

Robust Risk Budgeting with Independent Alpha Sources

1. Revisit and expand Blitz and Hottinga's assumptions
   1. The alpha portfolios are still independent of the beta portfolio and of each other
   2. The i^{th} alpha portfolio has an IR of IR_i, a TE of TE_i, and an Expected Shortfall of ES_i

2. Redefine IR in terms of tail risk: $IR'_i = ER_i / ES_i = IR_i \times TE_i / ES_i = IR_i \times TEES_i$

3. Modify Blitz and Hottinga's risk budgeting solution by replacing *IR*'s with modified *IR*'s

$$Risk\ Budget_i = \frac{IR_i^{'}}{\sqrt{IR_1^{'2} + \cdots IR_N^{'2}}} \times TE_{\text{Target}}$$

4. We continue to achieve our TE target, but underweight alpha sources with high tail risk!
   1. Relative allocation between alpha sources changes in accordance with the ratio of vol to tail risk
   2. But this is exactly what a conscientious portfolio manager does in practice!
   3. Important: For normally distributions, this solution preserves its mean variance optimality

Robust Risk Budgeting - Example

- 2 alpha sources, each with an *IR* of 1, TE_{Target}= 100 bp
  - Alpha Source 1: IR_1 =1, *TE* = 100 bp, *ES* = 200 bp, IR'_1 =1 x 100 / 200 =.50
  - Alpha Source 2: IR_1 =1, *TE* = 100 bp, *ES* = 300 bp, IR'_2 =1 x 100 / 300 =.33

- Blitz and Hottinga would allocate both strategies the same risk budget (71 bp)

- Our risk budget for alpha source 1 = $\frac{0.5}{\sqrt{0.5^2 + 0.33^2}} \times 100$ = 83 bp of TE

- Our risk budget for alpha source 2 = $\frac{0.33}{\sqrt{0.5^2 + 0.33^2}} \times 100$ = 55 bp of TE

- TE is still 100 bp, but we overallocate to the alpha source with lower tail risk

Robust Risk Budgeting – An Important Special Case

- If the alpha sources have the same IR, risk budgets are proportional to TEES

$$Risk\ Budget_i = \frac{TEES_i}{\sqrt{TEES_1^2 + \cdots + TEES_N^2}} \times TE_{\text{Target}}$$

- If all strategies have the same TEES, their risk budgets are equal

- This is as simple as it gets!
  - All risk budgets are naturally positive, no need for long-only constraints

- Our method has something in common with the Black-Litterman method
  - Both start with a simple optimal solution, then modify it in accordance with some auxiliary information
  - Black and Litterman modify their simple solution in accordance with auxiliary views on *ER*
  - We modify our simple optimal solution in accordance with auxiliary views on *ES*
  - They optimize the change to their initial allocation, we don't (and can't – the math is too hard)

Dealing With Correlations

- Berkelaar, Kobor and Tsumagiri: risk budgeting for correlated alpha sources

$$[Risk\ Budget]_i = \frac{\left[[\rho]^{-1}[IR]\right]_i}{\sqrt{[IR]^T[\rho]^{-1}[IR]}} \times TE_{\text{Target}}$$

- Have to invert the correlation matrix, cannot guarantee positive risk budgets

- Some pragmatic choices result in positive robust risk budgets with correlated alphas
  - Shrink correlation matrix, replace individual correlations by average correlation
    - Elton and Gruber (1973), Elton, Gruber and Urich (1978)
  - Replace all *IR*s by modified *IR*s

$$[Risk\ Budget]_i = \frac{\left[[\rho_{Average}]^{-1}[IR']\right]_i}{\sqrt{[IR']^T[\rho_{Average}]^{-1}[IR']}} \times TE_{\text{Target}}$$

The New Correlation Matrix and its Inverse

- Shrinking correlation matrices works well – no need for a long only constraint

- Replacing all correlations with the average correlation gives us a simple inverse

$$\begin{bmatrix} 1 & \bar{\rho} & . & \bar{\rho} \\ \bar{\rho} & . & . & . \\ . & . & . & \bar{\rho} \\ \bar{\rho} & . & \bar{\rho} & 1 \end{bmatrix}^{-1} = \frac{1}{(1-\bar{\rho})(1+(N-1)\bar{\rho})} \begin{bmatrix} 1+(N-2)\bar{\rho} & -\bar{\rho} & . & -\bar{\rho} \\ -\bar{\rho} & . & . & . \\ . & . & . & -\bar{\rho} \\ -\bar{\rho} & . & -\bar{\rho} & 1+(N-2)\bar{\rho} \end{bmatrix}$$

- Can use this to get a closed form MV risk budget in the equal IR case: $\dfrac{TE_{\text{Target}}}{\sqrt{N(1+(N-1)\bar{\rho})}}$

- When $\bar{\rho} \to 0,$ allocations $\to \dfrac{TE_{\text{Target}}}{\sqrt{N}}$ when $\bar{\rho} \to 1,$ allocations $\to \dfrac{TE_{\text{Target}}}{N}$

Example: Four Alpha Strategies

- Simulate portfolio returns (e.g.using RiskMetrics) or get time series of past returns
  - Compute Tracking Error and Expected Shortfall
  - Assume all alpha sources have the same IR

- Compute a risk budget (as a % of target vol)

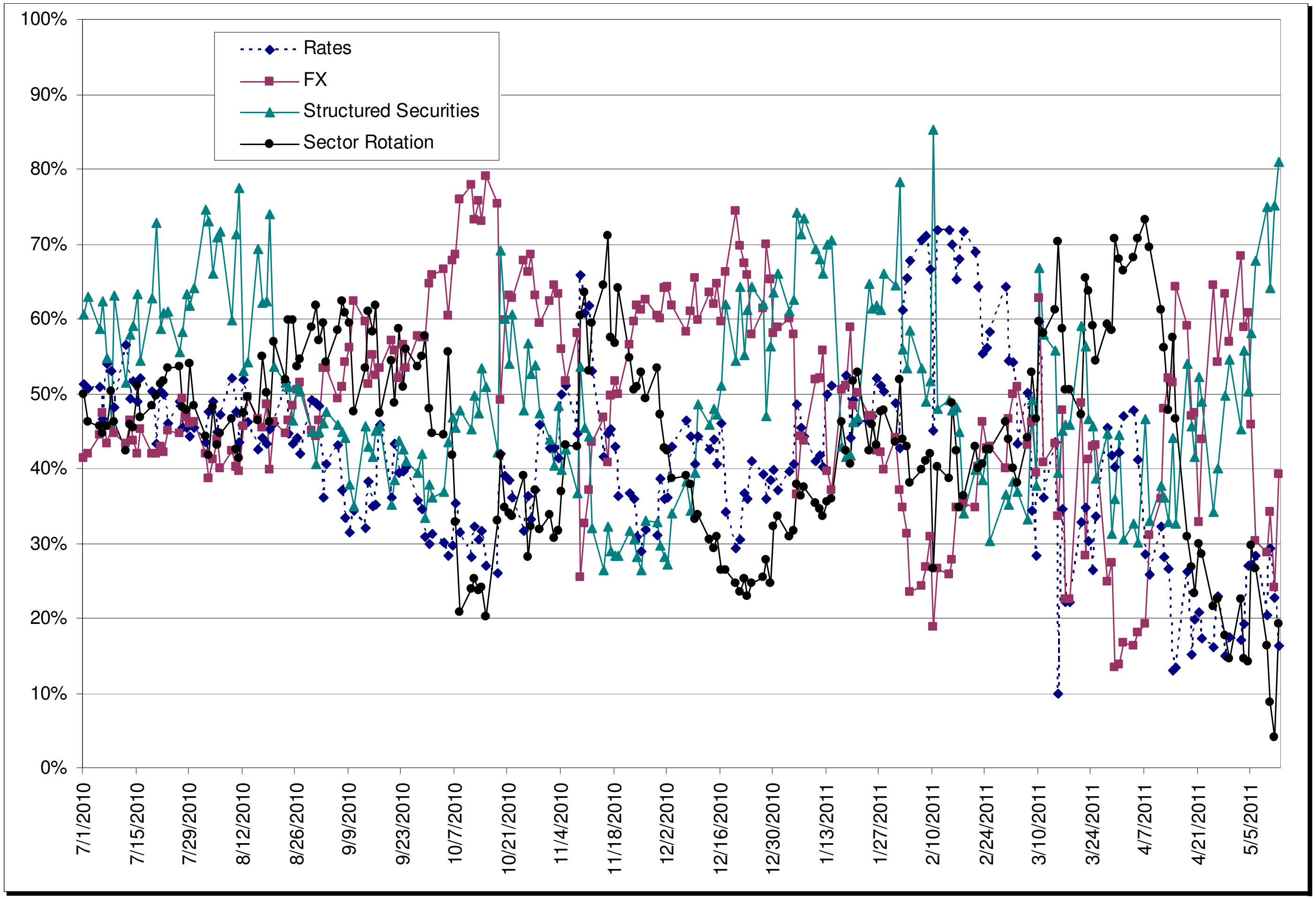
	Foreign Exchange	*Sector Rotation*	*Structured Securities*	*Interest Rates*
Tracking Error (1 day, bp)	1.78	1.00	1.82	0.95
Expected Shortfall (99% 1 day, bp)	7.10	2.40	4.55	3.61
TEES Ratio	0.25	0.42	0.40	0.26
Mean-Variance Allocation of Tracking Error	69%	58%	66%	15%
Allocation of Tracking Error using TEES	36%	61%	59%	38%

- Results

	Mean-Variance Risk Budget	*Risk Budget Using TEES*
Tracking Error (1 day, bp)	2.5	2.4
Expected Shortfall (1 day, bp)	8.4	7.1
TEES (TE / ES)	0.30	0.34

Allocating Risk Using TEES From a Simulation - II

- Allocations are positive, stable over time, data errors are rapidly corrected



Source: BNP Paribas Investment Partners

Summary and Open Questions

- Remarkably simple risk budgeting algorithm, can be solved on the back of an envelope
  - Gives excellent results in spite of its simplicity
    - 15% or greater reduction in ES at our target level of tracking error
  - Intuitive: users can identify overweights and underweights at a glance
  - Shrinking the covariance matrix generates positive portfolios without a long only constraint

- Open questions:
  - Is there a family of optimal algorithms to minimize ES under a TE constraint?
    - Algorithms ought to be indexed by the amount of information they require
  - How far is our solution from the true optimal solution?
  - Is there an easy way to get closer to the true optimum?

Appendix 1. Better Coherent Risk Measures

- Observation: Expected Shortfall is a function of the market environment

- Compute ES is many states of the world, take the average or the max
  - **M**ultiple **E**nvironment **A**verage / **M**aximum **E**xpected **S**hortfall

$$MEAES_{\alpha}(X) = E_S\left\{ES_{\alpha}\left(X^{s}\right)\right\}$$

$$MEMES_{\alpha}(X) = \max_{s \in S}\left\{ES_{\alpha}\left(X^{s}\right)\right\}$$

- Can show that these risk measures are coherent as well

- Generalizes Expected Shortfall in a natural way, gives us a much better picture of risk
  - Gives us insights into the particular states of the world that will bring us harm
  - *MEAES* is better suited to unlevered portfolios (asset management)
  - *MEMES* is better suited to levered portfolios (market marking, hedge funds)

Appendix 2. Robust Instantaneous Average Correlations

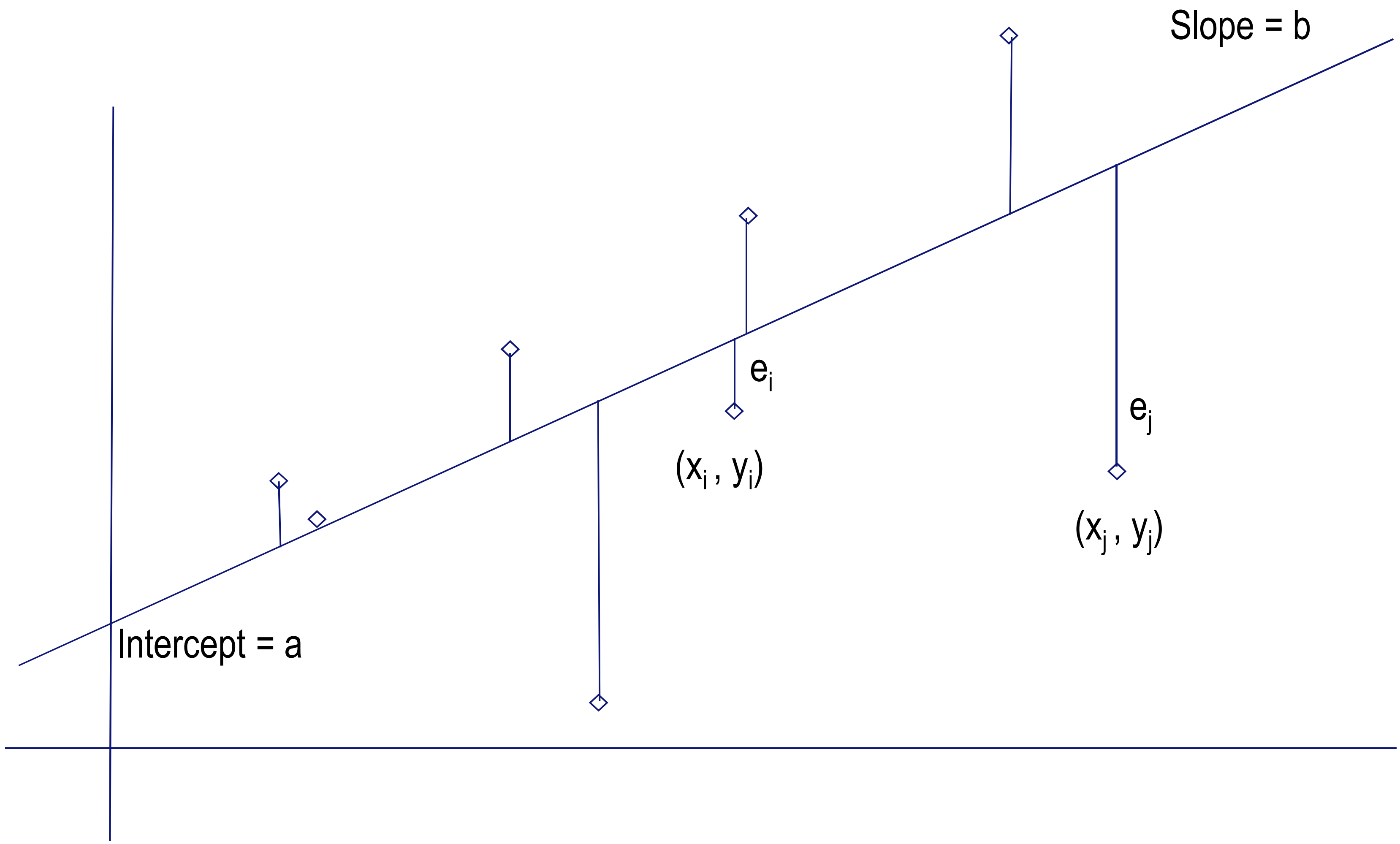
- Given a cross-sectional set of returns for *N* zero mean multivariate normal random variables, find a robust estimate of their instantaneous AVERAGE correlation
  - Count the number with a positive return
  - Count the number with a negative return

$$\hat{\rho}_t \equiv 1 - 4 \times \frac{N_t^+ \times N_t^-}{N(N-1)}$$

- Proof follows from the fact that for bi-variate normals, $P\left[r_{it} \times r_{jt} < 0\right] = \dfrac{\cos^{-1} \rho_{ij}}{\pi}$
- For any particular pair of random variables, $\hat{\rho}_{ijt} = 2 \times u(r_{it} \times r_{jt}) - 1$
- Average over all pairs of securities and simplify to get the result
- Robust to outliers, works well even if the returns are not normally distributed
  - Can exponentially weight to get dynamic estimates of average correlation

Appendix 3. OLS Revisited

- Classic explanation of OLS – choose a and b to minimize the sum of squared errors

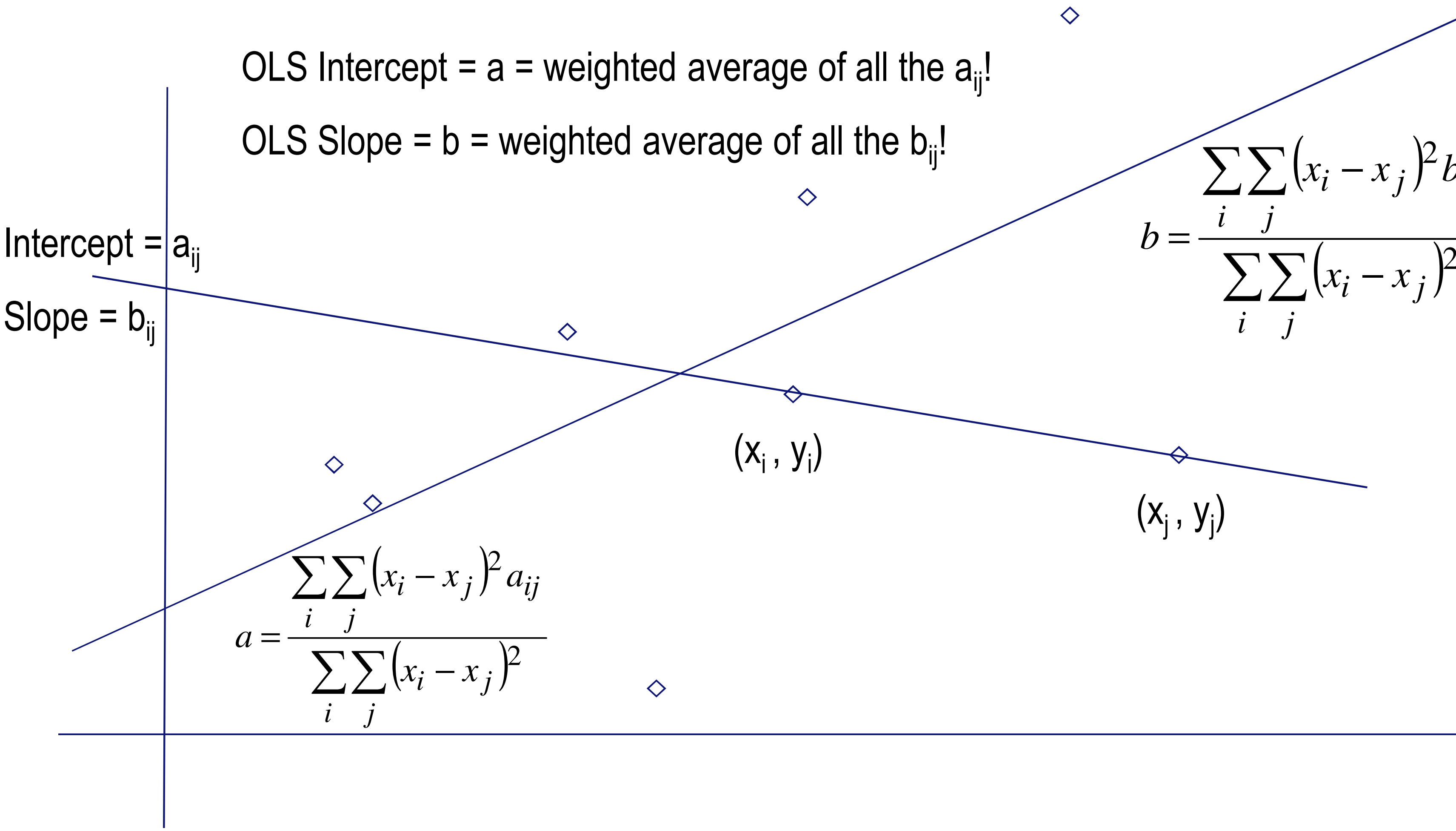


Appendix 3. OLS Revisited

- A different explanation of OLS – form a weighted average of all possible slopes

OLS Intercept = a = weighted average of all the a_{ij}!

OLS Slope = b = weighted average of all the b_{ij}!



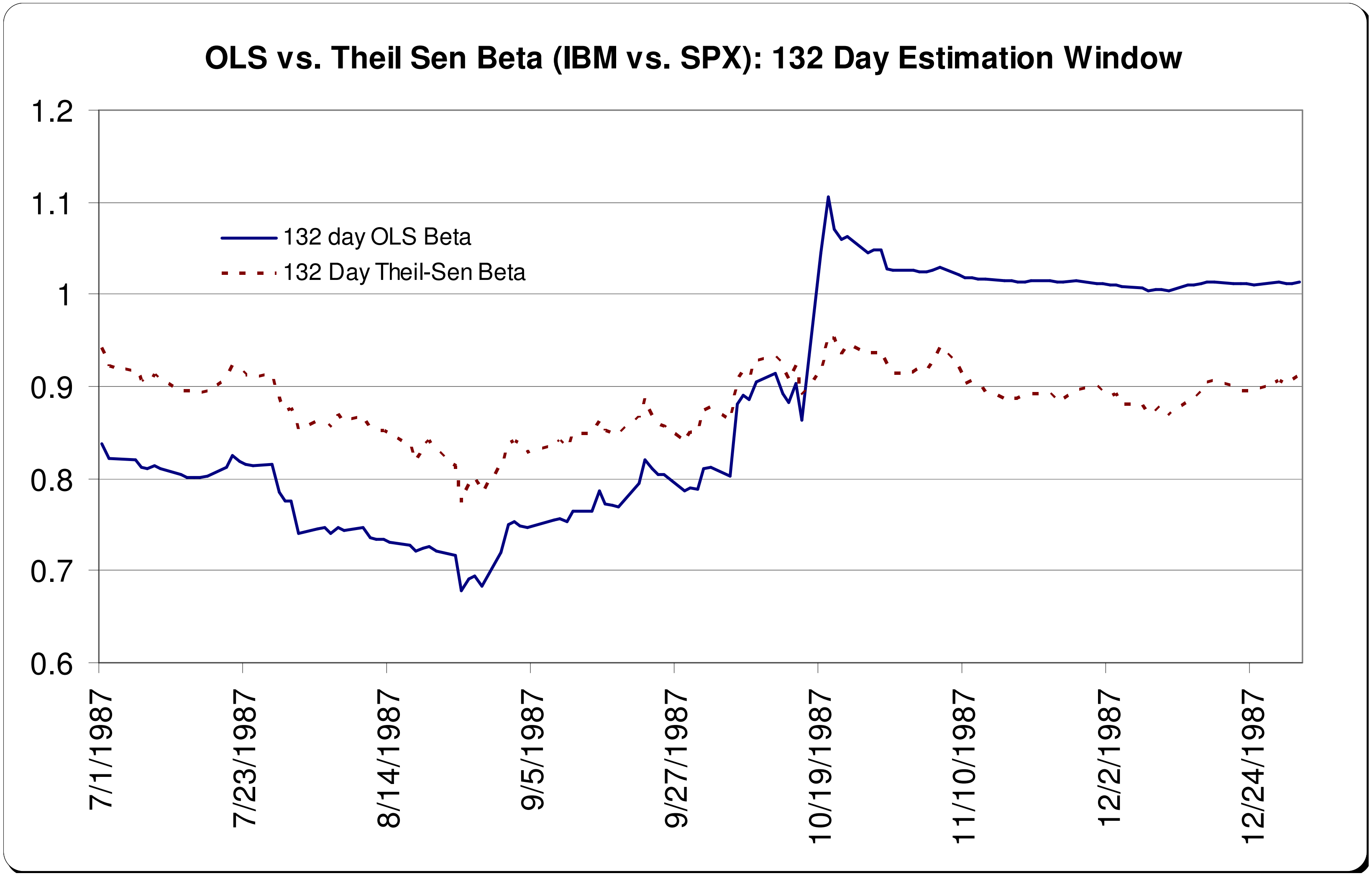
$$b = \frac{\sum_i \sum_j (x_i - x_j)^2 b_{ij}}{\sum_i \sum_j (x_i - x_j)^2}$$

$$a = \frac{\sum_i \sum_j (x_i - x_j)^2 a_{ij}}{\sum_i \sum_j (x_i - x_j)^2}$$

Appendix 3. The Theil-Sen Estimator

- Theil-Sen robust estimate of slope = median of all N(N-1)/2 slopes (i.e. the b_{ij}'s)
- Theil-Sen intercept chosen to make the median error 0
- Incredibly robust to outliers, works well even if the returns are not normally distributed
  - Approximately 25% of the points can be arbitrarily bad before it stops working
  - OLS, in contrast can be corrupted by a single outlier
  - Sum of squared errors is always within 10% or so of OLS
- In the univariate case, Theil-Sen should replace OLS universally
  - Not a good choice for multivariate regression – Theil Sen can converge to the wrong answer
    - Particularly true if the relationship is nonlinear
  - But there are other robust regression algorithms that work well with multivariate data

Estimating IBM's Beta Around the Crash of 1987



Source: BNP Paribas Investment Partners

Appendix 4. Robust Estimates of correlation

$$
\begin{aligned}
\rho_{xy} &= \frac{Cov(x,y)}{\sigma_x \times \sigma_y} \\
&= \sqrt{\frac{Cov(x,y)^2}{\sigma_x^2 \times \sigma_y^2}} \\
&= \sqrt{\beta_{x|y} \times \beta_{y|x}}
\end{aligned}
$$

- Correlation is a geometric mean of two betas!
- UseTheil-Sen twice to estimate both robust betas, then compute a robust correlation
- Very effective in practice –
  - Filters noise beautifully
  - Essentially no give up in efficiency to the maximum likelihood estimator

Simulations With Independent Random Variables

Distribution= Normal(0,1), N=100									
Percentiles	**1**	**5**	**10**	**25**	**50**	**75**	**90**	**95**	**99**
Theil Sen Slope	-0.26	-0.17	-0.14	-0.07	0	0.07	0.13	0.18	0.26
Least Squares Slope	-0.24	-0.17	-0.13	-0.07	0	0.07	0.13	0.17	0.24
Theil Sen Intercept	-0.3	-0.21	-0.16	-0.09	0	0.08	0.16	0.21	0.3
Least Squares Intercept	-0.23	-0.17	-0.13	-0.07	0	0.07	0.13	0.17	0.23
Theil Sen Mean Square Error	0.69	0.77	0.81	0.89	0.98	1.08	1.17	1.24	1.35
Least Squares Mean Square Error	0.69	0.76	0.81	0.88	0.98	1.07	1.16	1.23	1.33
Theil Sen Median Error	0	0	0	0	0	0	0	0	0
Least Squares Median Error	-0.17	-0.12	-0.09	-0.05	0	0.05	0.09	0.12	0.18
Theil Sen correlation	-0.24	-0.17	-0.14	-0.07	0	0.07	0.13	0.18	0.25
Least Squares correlation	-0.23	-0.17	-0.13	-0.07	0	0.07	0.13	0.16	0.23

Distribution= Pareto(2), N=100									
Percentiles	**1**	**5**	**10**	**25**	**50**	**75**	**90**	**95**	**99**
Theil Sen Slope	-0.12	-0.07	-0.05	-0.03	0	0.03	0.06	0.09	0.14
Least Squares Slope	-0.35	-0.18	-0.13	-0.07	-0.02	0.03	0.13	0.23	0.65
Theil Sen Intercept	1.17	1.24	1.28	1.34	1.41	1.49	1.57	1.62	1.74
Least Squares Intercept	0.98	1.49	1.62	1.78	1.95	2.16	2.41	2.62	3.44
Theil Sen Mean Square Error	0.59	0.88	1.11	1.67	2.83	5.46	12.29	22.67	107.54
Least Squares Mean Square Error	0.52	0.78	0.98	1.47	2.53	4.97	11.54	21.47	104.8
Theil Sen Median Error	0	0	0	0	0	0	0	0	0
Least Squares Median Error	-1.51	-0.97	-0.83	-0.64	-0.5	-0.4	-0.32	-0.28	-0.22
Theil Sen correlation	-0.11	-0.07	-0.05	-0.03	0	0.03	0.06	0.08	0.13
Least Squares correlation	-0.15	-0.11	-0.1	-0.06	-0.02	0.04	0.12	0.19	0.37

Appendix 5a. Estimating Robust Correlation Matrices

- Create a robust correlation matrix using N(N-1)/2 Theil-Sen correlations
- Use Higham's projection algorithm to make it non-negative definite

Algorithm H (Higham (2002))

1. $\Delta S_0 = 0,\ X_0 = I,\ Y_0 = \hat{\rho},\ k = 0$
2. While $\| Y_k - X_k \| > \varepsilon$, Do
   a. $k=k+1$
   b. $R_k = Y_{k-1} - \Delta S_{k-1}$ (Dykstra's correction to speed convergence)
   c. $X_k = P_S(R_k)$
   d. $\Delta S_k = X_k - R_k$
   e. $Y_k = P_U(X_k)$
3. $\rho^* = Y_k$

Appendix 5b. Estimating Robust Covariance Matrices

- Start with a robust non-negative definite correlation matrix ρ^*
- Get robust estimators of vol e.g. Rousseeuw and Croux's Q_N
  - $Q_N(X) = 1.19 \times 25^{th}$ percentile of $\{|x_i - x_j|, i < j\}$.
- $$\hat{C} = \Sigma \rho^* \Sigma, \quad \Sigma = \begin{bmatrix} \sigma_1^{Robust} & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & 0 \\ 0. & . & 0 & \sigma_N^{Robust} \end{bmatrix}$$

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