A Critical Review of Correlation-based Measures of Portfolio Diversification

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The Portfolio Diversification Index (PDI) and the Diversification Ratio (DR) both aim to (1) quantify diversification characteristics expressly related to correlations and (2) provide frameworks for constructing diversified portfolios.

The PDI and DR are in fact closely related to two very well-known risk-based portfolio construction approaches: Minimum Variance Portfolios (MVP) and Risk Parity Portfolios (RPP).

- We introduce the Minimum Correlation Portfolio (MCP) and show how it can be used to solve for the maximum value of the DR.
- We clarify the properties of the PDI and show that it quantifies diversification characteristics specific to so-called “Naïve Risk Parity” portfolios.
- We highlight a significant weakness in the PDI and show that it can generate misleading estimates of diversification when there are negative correlations among assets, in contrast to the DR which clearly distinguishes the effects of positive and negative correlations on portfolio diversification.
Diversification and risk-based portfolio construction

- Risk-based portfolio construction approaches grew in popularity following the 2008 global financial crisis, when many portfolios and investment strategies thought to be well-diversified became spectacularly and dramatically undiversified.

- Risk-based methods eschew expected returns and focus solely on volatilities and correlations, with the goal of producing portfolios with “better” diversification characteristics vis-à-vis mean-variance optimal (MVO) portfolios.

- Several approaches to measuring diversification and constructing diversified portfolios have been proposed, which raises an important question:

  *What exactly do we mean by diversification?*
Defining and measuring diversification

- “Diversification strives to smooth out unsystematic risk events in a portfolio so that the positive performance of some investments will neutralize the negative performance of others. Therefore, the benefits of diversification will hold only if the securities in the portfolio are not perfectly correlated.” (investopedia.com)

- “A well-diversified portfolio is one that is expected to be immune against shocks created by a single or a few assets.” (Meucci, 2009; Frahm and Weichers, 2011)

- Market portfolio; equal-weighted portfolio (“naïve” diversification)
- Number of assets; portfolio weights; breadth of positions
- Information entropy
- Minimum variance and risk parity portfolios
- Low co-movement across assets (“most diversified portfolio”)
- No definitive definition or unique measure of “diversification”
Defining and calculating portfolio risk

- Define the NxN covariance matrix of asset excess returns, \( V \):

\[
V = SCS = \begin{pmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_N
\end{pmatrix}
\begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1N} \\
\rho_{12} & 1 & \cdots & \rho_{2N} \\
\vdots & & \ddots & \vdots \\
\rho_{1N} & \rho_{2N} & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_N
\end{pmatrix}
= \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\
\vdots & & \ddots & \vdots \\
\sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN}
\end{pmatrix}
\]

- Define \( V(w) \), the weighted covariance matrix associated with a portfolio that has an Nx1 vector of portfolio weights, \( w \):

\[
V(w) = WVW = \begin{pmatrix}
w_1 & 0 & \cdots & 0 \\
0 & w_2 & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & w_N
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\
\vdots & & \ddots & \vdots \\
\sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN}
\end{pmatrix}
\begin{pmatrix}
w_1 & 0 & \cdots & 0 \\
0 & w_2 & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & w_N
\end{pmatrix}
= \begin{pmatrix}
w_1^2 \sigma_{11} & w_1 w_2 \sigma_{12} & \cdots & w_1 w_N \sigma_{1N} \\
w_2 w_1 \sigma_{21} & w_2^2 \sigma_{22} & \cdots & w_2 w_N \sigma_{2N} \\
\vdots & & \ddots & \vdots \\
w_N w_1 \sigma_{N1} & w_N w_2 \sigma_{N2} & \cdots & w_N^2 \sigma_{NN}
\end{pmatrix}
\]

- The variance of this portfolio is

\[
\sigma^2(w) = l' V(w) l = w' V w = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \quad (l \text{ is an Nx1 vector of ones})
\]
Contributions to portfolio risk

- **Marginal contributions to portfolio risk (MCR):**

  \[ MCR(w) = \frac{\partial \sigma(w)}{\partial w'} = \frac{V_w}{\sigma(w)} \]

  - The MCR of an asset is the approximate increase or decrease in portfolio risk when the weight of that asset is increased by one percentage point, where the increase is assumed to be financed using cash (as opposed to selling other assets in the portfolio).

- **Total contributions to portfolio risk (TCR):**

  \[ TCR(w) = W \left[ \frac{\partial \sigma(w)}{\partial w'} \right] = \frac{WV_w}{\sigma(w)} \]

  - TCRs are the amounts of portfolio risk contributed by each asset. The sum of the TCRs is equal to the volatility of the portfolio:

  \[ \sum_{i=1}^{N} TCR_i(w) = \frac{l'WV_w}{\sigma(w)} = \frac{w'V_w}{\sigma(w)} = \sigma(w) \]
Minimum Variance Portfolio

- **Minimum Variance Portfolio (MVP):**

  \[
  \min_w \frac{1}{2} w'Vw + (1 - w'l) \cdot \theta
  \]

  \[\Rightarrow w_{MVP} = (l'V^{-1}l)^{-1}V^{-1}l, \quad \sigma^2(w_{MVP}) = w_{MVP}'Vw_{MVP} = (l'V^{-1}l)^{-1}\]

- The MVP has the lowest possible variance out of all fully invested portfolios.

- A key feature of the MVP is that the marginal contributions to risk are the same for all assets:

  \[
  MCR(w_{MVP}) = \frac{Vw_{MVP}}{\sigma(w_{MVP})} = l\sigma(w_{MVP})
  \]

- **The MVP is diversified in terms of marginal contributions to risk:** an incremental addition to the weight of an asset will increase the risk of the MVP by the same quantity as an identical incremental addition to the weight of any other asset.

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Risk Parity Portfolios (RPP) are characterized by equal total contributions to portfolio risk from each asset:

\[ TCR_i \left( w_{RPP} \right) = TCR_j \left( w_{RPP} \right) \quad \forall \ i, j \]

A RPP is diversified in terms of total contributions to risk: each asset contributes an equal amount to the volatility of the portfolio, and therefore gains and losses in the portfolio will not be dominated by an individual position in any asset.

- There is no closed-form solution to equalizing total risk contributions across assets unless all correlations are identical.
- In the case of identical correlations, RPP weights are simply equal to the reciprocal of each asset’s volatility (normalized to sum to one). This inverse-volatility weighting scheme is commonly referred to as “Naïve Risk Parity” (NRP), and is often used in practice to approximate a RPP:

\[ w_{NRP} = (l'S^{-1}l)^{-1} S^{-1}l \]
Portfolio Diversification Index (PDI)

- Rudin and Morgan (2006) apply principal component analysis (PCA) to a correlation matrix to calculate the Portfolio Diversification Index (PDI)
  - PCA transforms N correlated assets into a set of N uncorrelated principal components (PCs) that are ordered according to how much variation in the returns is retained by each PC

\[ PDI = 2 \cdot \sum_{i=1}^{N} i \cdot RS_i - 1, \quad RS_i = \frac{\lambda_i}{\sum_{j=1}^{N} \lambda_j} \]

- Interpretation is that the set of N correlated assets offers the same degree of diversification as PDI uncorrelated assets:
  - The PDI is bound between one and N
  - PDI = N if all assets are perfectly uncorrelated ("ideal" diversification)
  - PDI < N reflects more extensive co-movement across assets; more return variation is explained by the first few PCs
  - PDI \( \sim \) 1 indicates diversification is effectively impossible

\( \lambda_i \) = eigenvalue associated with the \( i \)th PC
RS\( _i \) = relative strength of the \( i \)th PC
PDI and Risk Parity

- The PDI provides a summary statistic of diversification distinctly related to correlations as well as a criterion for portfolio construction.

- Implicit in using a correlation matrix is that volatilities have been standardized to the same level → the PDI specifically quantifies the diversification characteristics of a Naïve Risk Parity portfolio:

\[
V(w_{\text{NRP}}) = W_{\text{NRP}} V W_{\text{NRP}} = \begin{pmatrix}
\sigma_1^{-1} & 0 & \cdots & 0 \\
0 & \sigma_2^{-1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_N^{-1}
\end{pmatrix} \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\
\sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\
\vdots & \ddots & \ddots & \vdots \\
\sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN}
\end{pmatrix} \begin{pmatrix}
\sigma_1^{-1} & 0 & \cdots & 0 \\
0 & \sigma_2^{-1} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_N^{-1}
\end{pmatrix} = \begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1N} \\
\rho_{12} & 1 & \cdots & \rho_{2N} \\
\vdots & \ddots & \ddots & \vdots \\
\rho_{1N} & \rho_{2N} & \cdots & 1
\end{pmatrix} = C
\]

- The PDI does not measure the diversification of a given portfolio per se, but rather the diversification potential of a set of assets were they to be combined into a NRP portfolio.

- Equal-weighted portfolios constructed using the PDI will not reflect the NRP diversification characteristics being measured by the PDI.
Choueifaty and Coignard (2008) quantify diversifying properties associated with correlations using the Diversification Ratio (DR):

$$DR(w) = \frac{\sqrt{w'V_Lw}}{\sqrt{w'Vw}} = \frac{w's}{\sigma(w)}$$,  \( V = SCS \),  \( V_L = SLS \),  \( L = ll' \),  \( s = Sl \)

The DR is the volatility of a given portfolio if all correlations were equal to one divided by the volatility of the portfolio calculated using prevailing correlation estimates.

- The DR is bounded from below by one.
- It is unbounded from above, with higher values reflecting the diversifying effects of relatively low correlations across assets.
- \( DR = \sqrt{N} \) if all assets are perfectly uncorrelated ("ideal" diversification).

Unlike the PDI, the DR can be used to measure the diversification characteristics of a variety of portfolios.
The Most Diversified Portfolio (MDP) maximizes the DR

The maximized DR quantifies the ultimate diversification potential of the assets that comprise a given portfolio or pool of securities, where diversification is now defined as the lowest possible correlation across assets.

Solving for the MDP:

\[
\min_w \frac{1}{2} w'Vw + (1 - w's) \cdot \theta, \quad s = Sl
\]

\[
\rightarrow w_{MDP} = (s'V^{-1}s)^{-1}V^{-1}s = (l'C^{-1}l)^{-1}S^{-1}C^{-1}l
\]

\[
\sigma^2(w_{MDP}) = w_{MDP}'Vw_{MDP} = (s'V^{-1}s)^{-1} = (l'C^{-1}l)^{-1}
\]

The maximized value of the DR is

\[
DR_{MAX} = DR(w_{MDP}) = \sqrt{w_{MDP}'Vlw_{MDP}} \div \sqrt{w_{MDP}'Vw_{MDP}} = \frac{\sqrt{(l'C^{-1}l)^{-1}l'C^{-1}l}}{(l'C^{-1}l)^{-\frac{1}{2}}} = \frac{1}{\sigma(w_{MDP})}
\]
Normalizing volatilities to equal one and solving for the MVP or the MDP produces the minimum correlation portfolio (MCP):

$$\min_w \frac{1}{2} w' C w + (1 - w'l) \cdot \theta$$

$$\Rightarrow w_{MCP} = (l'C^{-1}l)^{-1} C^{-1}l, \quad \tilde{\sigma}^2(w_{MCP}) = w_{MCP}' C w_{MCP} = \left(l'C^{-1}l\right)^{-1} = \sigma^2(w_{MDP})$$

The normalized variance of the MCP is equal to the variance of the MDP

$\Rightarrow$ MDP variance has the lowest possible correlation across assets

Alternate derivation of the MDP:

1. Standardize volatilities ("Choueifaty Synthetic Asset Transformation") and solve for the MCP weights

2. Multiply the MCP weights by NRP weights ("Choueifaty Synthetic Asset Back-Transformation"):

$$w_{MDP} = \left(s' V^{-1} s\right)^{-1} V^{-1} s = \left(l'C^{-1}l\right)^{-1} S^{-1} C^{-1}l = S^{-1} w_{MCP}$$
The inverse of the square of the maximized DR is equal to the variance of the MDP:

$$\sigma^2(w_{MDP}) = \frac{1}{DR_{MAX}^2} = (l'C^{-1}l)^{-1}$$

The variance of the MDP (VMDP) has convenient properties that make it a useful measure of diversification characteristics associated with correlations:

- It is bound between zero and one
- \(1/DR_{MAX}^2 = 1/N\) when all correlations equal zero ("ideal" diversification)
- \(1/DR_{MAX}^2 > 1/N\) indicates less diversification potential vis-à-vis ideal diversification; \(1/DR_{MAX}^2 \approx 1\) signifies no possibility to diversify
- \(1/DR_{MAX}^2 < 1/N\) reflects hedging potential associated with negative correlations; \(1/DR_{MAX}^2 \approx 0\) indicates the existence of assets that are nearly perfectly negatively correlated

**From here on out we will use 1/DR² as our preferred diversification measure**
The DR and PDI both quantify diversification in a single number; however, the results can be conflicting.

Example: scenario analysis for a global asset allocation manager using correlations and volatilities that are expected to prevail during periods of market turbulence, with a particular interest in how diversification opportunities are likely to be affected.

### Normal Markets

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>Int'l Developed</th>
<th>Emerging Markets</th>
<th>Global Bonds</th>
<th>Volatilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td>1</td>
<td>60%</td>
<td>50%</td>
<td>40%</td>
<td>15%</td>
</tr>
<tr>
<td>Int'l Developed</td>
<td>60%</td>
<td>1</td>
<td>70%</td>
<td>30%</td>
<td>18%</td>
</tr>
<tr>
<td>Emerging Markets</td>
<td>50%</td>
<td>70%</td>
<td>40%</td>
<td>1</td>
<td>22%</td>
</tr>
<tr>
<td>Global Bonds</td>
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<td>30%</td>
<td>40%</td>
<td>8%</td>
<td>8%</td>
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</table>

### Turbulent Markets

<table>
<thead>
<tr>
<th></th>
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<th>Emerging Markets</th>
<th>Global Bonds</th>
<th>Volatilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td>1</td>
<td>90%</td>
<td>80%</td>
<td>-10%</td>
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</tr>
<tr>
<td>Int'l Developed</td>
<td>90%</td>
<td>1</td>
<td>85%</td>
<td>-10%</td>
<td>36%</td>
</tr>
<tr>
<td>Emerging Markets</td>
<td>80%</td>
<td>85%</td>
<td>1</td>
<td>-10%</td>
<td>44%</td>
</tr>
<tr>
<td>Global Bonds</td>
<td>-10%</td>
<td>-10%</td>
<td>-10%</td>
<td>1</td>
<td>12%</td>
</tr>
</tbody>
</table>
The DR indicates diversification potential is greater during turbulent periods for both the MDP and NRP.

MDP and NRP weights show that diversification is increased by reducing exposure to equities and adding to bonds.
- The PDI indicates diversification opportunities are lower during times of turbulence
- The first two PCs account for much more variation in turbulent periods, which is not surprising given the increase in stock market correlations
Comparing the DR and PDI (con’t)

- This example highlights a critical difference in these approaches to measuring diversification:
  - The PDI accurately reflects the fact that more substantial co-movement exists among the majority of assets (i.e., the equity markets) during times of turbulence
  - The maximized DR reflects the increase in diversification potential from the standpoint of a portfolio that is optimized to have the lowest possible correlation across assets
- A stronger degree of positive correlation across most of the assets does not necessarily imply an overall reduction in the potential to diversify
- *The PDI may belie portfolio diversification opportunities that are more readily apparent from the perspective of the DR/MDP*
Negative correlations and hedging characteristics

- Negative correlations are problematic for the PDI:
  - The PDI is symmetric with respect to positive and negative correlations
  - This is because the eigenvalues are the same regardless of the sign of $\rho$

\[
\frac{1}{DR_{\text{MAX}}^2} = \frac{1 + \rho}{2}
\]

\[
PDI = 2 - |\rho|
\]
Example: three assets

- The two-asset example provides a simple illustration of the properties of both diversification measures; visualizing these properties becomes more complicated when \(N > 2\):
  - Higher dimensionality limits the number of possible assets
  - The range of possible values for the correlation coefficients must be restricted in order to maintain a positive semi-definite correlation matrix
- With three assets a positive semi-definite matrix is guaranteed by defining one correlation as a function of (1) the other two correlations, and (2) a partial correlation coefficient (PCC) that holds fixed the asset the other two correlations have in common:

\[
\rho_{23} = \rho_{12} \cdot \rho_{13} + \sqrt{1 - \rho_{12}^2} \cdot \sqrt{1 - \rho_{13}^2} \cdot \rho_{23|1}
\]

- Given a fixed value of the PCC, the diversification measures can be calculated for the entire constrained range of 3x3 correlation matrices and plotted as functions of the two free correlation coefficients
Conditional Correlations and Diversification/Hedging Profiles, $\rho_{23|1} = -0.99$
Conditional Correlations and Diversification/Hedging Profiles, $\rho_{23|1} = -0.75$
Conditional Correlations and Diversification/Hedging Profiles, $\rho_{23|1} = 0$

- $\rho_{23} = f(\rho_{12}, \rho_{13}; \rho_{23|1} = 0)$
- MDP, long-only
- MDP, unconstrained
- $1/\text{PDI}$
Conditional Correlations and Diversification/Hedging Profiles, \( \rho_{23|1} = 0.75 \)
Conditional Correlations and Diversification/Hedging Profiles, $\rho_{23|1} = 0.99$
Comparing the MDP and PDI

- The maximized DR profiles exhibit a number of distinctive characteristics:
  - $1/\text{DR}^2_{\text{MAX}}$ is close to its maximum value of one only when all of the correlations are nearly equal to one → *low diversification potential is due solely to relatively high positive correlations across all of the assets*
  - $1/\text{DR}^2_{\text{MAX}}$ is close to zero when negative correlations become relatively extreme → *significant hedging potential exists whenever two of the assets are close to being perfectly negatively correlated*
  - Constraining the MDP weights to be non-negative reduces diversification opportunities but does not affect the fundamental properties of the maximized DR
  - The no-shorting constraint also reflects more realistic investment possibilities as unconstrained MDP weights can take on large values that would require extreme leverage to implement
The profiles of the inverted PDI differ considerably from those of the MDP:

- The values of the PDI are symmetric with respect to positive and negative correlations, with the inverse of the PDI close to one when all of the correlations are at extremes irrespective of sign.
- The inverse of the PDI reaches its lower bound of $1/N = 1/3$ when all three correlations are equal to zero, which occurs only in the case of $\rho_{23|1} = 0$.

*The PDI can be viewed as operating on the absolute values of the correlations, and thus can produce misleading estimates of diversification potential.*
Example: hedge fund investment styles

- We compare the DR and PDI using a universe consisting of the S&P 500 and nine Hedge Fund Research (HFR) investment style return indices:

<table>
<thead>
<tr>
<th>Investment Style</th>
<th>Return Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>Merger Arbitrage</td>
</tr>
<tr>
<td>Short Bias</td>
<td>Macro</td>
</tr>
<tr>
<td>Equity Market Neutral</td>
<td>FI Convertible Arbitrage</td>
</tr>
<tr>
<td>Quantitative Directional</td>
<td>FI Multi-Strategy</td>
</tr>
<tr>
<td>Distressed/Restructuring</td>
<td>FI Corporate</td>
</tr>
</tbody>
</table>
### Correlations of S&P 500 and Hedge Fund Style Monthly Returns, January 1990 - June 2014

<table>
<thead>
<tr>
<th>S&amp;P 500</th>
<th>Short Bias</th>
<th>Equity Market Neutral</th>
<th>Quantitative Directional</th>
<th>Distressed/Restructuring</th>
<th>Merger Arbitrage</th>
<th>Macro</th>
<th>FI Convert Arbitrage</th>
<th>FI Multi-Strategy</th>
<th>FI Corporate</th>
<th>Volatility (annualized)</th>
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</thead>
<tbody>
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<td></td>
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<td></td>
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<td>14.7%</td>
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<td>S&amp;P 500</td>
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<td>Distressed/Restructuring</td>
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<td>FI Multi-Strategy</td>
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<td>-46%</td>
<td>26%</td>
<td>60%</td>
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<td>57%</td>
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<td>67%</td>
<td>81%</td>
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<tr>
<td>FI Corporate</td>
<td>53%</td>
<td>-46%</td>
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<td>60%</td>
<td>83%</td>
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<td>38%</td>
<td>67%</td>
<td>81%</td>
<td>6.5%</td>
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### Ideal Diversification (1/N) & PDI

<table>
<thead>
<tr>
<th></th>
<th>No Short Bias</th>
<th>With Short Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ideal Diversification (1/N)</td>
<td>0.11</td>
<td>0.10</td>
</tr>
<tr>
<td>1/PDI</td>
<td>0.28</td>
<td>0.27</td>
</tr>
<tr>
<td>1/DR²_MAX</td>
<td>0.47</td>
<td>0.07</td>
</tr>
</tbody>
</table>

- **The maximized DR reflects the hedging potential afforded by the Short Bias Index**
- **The PDI indicates diversification potential is unchanged with the inclusion of Short Bias**
36-month rolling correlations and volatilities, December 1992 – June 2014

- The maximized DR quantifies the hedging potential introduced by the Short Bias index: $1/\text{DR}^2_{\text{MAX}}$ always lies below the ideal diversification value of $1/N$

- The PDI continually indicates varying degrees of less-than-ideal diversification potential for the universe that includes the Short Bias index
Methods that use the PDI as a tool to construct diversified portfolios have recently been introduced (for example, Crezee and Swinkels, 2010; Diyarbakirlioglu and Satman, 2014)

- The approaches involve iterative procedures that combine assets until the maximum PDI is found for a portfolio with a pre-specified number of holdings
- These maximized PDI rules are ubiquitously used to construct equally weighted portfolios, as opposed to Naïve Risk Parity portfolios

We use the maximum PDI to construct portfolios consisting of the S&P 500 combined with the hedge fund style indices, with and without the Short Bias Index

- Use full-sample volatilities and correlations
- Portfolios range from 2 assets to all 10 assets
- Compare equal-weighted portfolio, NRP portfolios and long-only MDPs
Equal-weighted portfolios

**Equal-weighted portfolio weights based on maximizing the PDI, no Short Bias**

<table>
<thead>
<tr>
<th>N</th>
<th>1/PDI</th>
<th>1/DR²</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.58</td>
<td>0.79</td>
<td>7.89</td>
</tr>
<tr>
<td>3</td>
<td>0.42</td>
<td>0.61</td>
<td>6.55</td>
</tr>
<tr>
<td>4</td>
<td>0.35</td>
<td>0.55</td>
<td>5.87</td>
</tr>
<tr>
<td>5</td>
<td>0.30</td>
<td>0.54</td>
<td>5.20</td>
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<td>0.28</td>
<td>0.54</td>
<td>5.14</td>
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<td>0.27</td>
<td>0.56</td>
<td>5.15</td>
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<tr>
<td>8</td>
<td>0.26</td>
<td>0.60</td>
<td>5.88</td>
</tr>
<tr>
<td>9</td>
<td>0.26</td>
<td>0.60</td>
<td>5.60</td>
</tr>
</tbody>
</table>

**Equal-weighted portfolio weights based on maximizing the PDI, with Short Bias**

<table>
<thead>
<tr>
<th>N</th>
<th>1/PDI</th>
<th>1/DR²</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.58</td>
<td>0.79</td>
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<td>0.35</td>
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<td>5.87</td>
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<tr>
<td>5</td>
<td>0.30</td>
<td>0.54</td>
<td>5.20</td>
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<tr>
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<td>0.28</td>
<td>0.13</td>
<td>3.19</td>
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<tr>
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<td>3.40</td>
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<tr>
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<td>0.26</td>
<td>0.19</td>
<td>3.44</td>
</tr>
<tr>
<td>10</td>
<td>0.27</td>
<td>0.22</td>
<td>3.93</td>
</tr>
</tbody>
</table>
Naïve Risk Parity (NRP) portfolios

Naïve Risk Parity portfolio weights based on maximizing the PDI, no Short Bias

Naïve Risk Parity portfolio weights based on maximizing the PDI, with Short Bias
Most Diversified Portfolios (MDP)

Most Diversified Portfolios, no Short Bias

Most Diversified Portfolios, with Short Bias

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Portfolio construction -- summary

- **Equal-weighted portfolios**
  - The Short Bias Index is not included for portfolios with less than 6 assets
  - The PDI indicates diversification is similar for portfolios with 5 or more assets
  - The DR and portfolio risk both fall significantly once the Short Bias Index is included

- **Naïve Risk Parity portfolios**
  - NRP weights differ considerably from equal weights
  - Risk is much lower for portfolios with 5 or fewer holdings, even though positions are more concentrated
  - Compared with equal weights, DRs are lower for portfolios with N < 6 but higher for 6 or more holdings when the Short Bias Index is included

- **Most Diversified Portfolios**
  - Long-only MDPs are very different from maximum PDI portfolios
  - Maximum diversification potential is reached with 6 assets ex-Short Bias and just 4 assets including Short Bias
  - All portfolios include the Short Bias Index when it is available for portfolio selection
Conclusion

- The PDI and the DR have become contenders in the quest for achieving diversification through risk-based portfolio construction methods.
- Both approaches can be interpreted in terms of Minimum Variance and Risk Parity investment strategies:
  - The PDI quantifies diversification specific to Naïve Risk Parity portfolios.
  - The maximized DR can be found using the Minimum Correlation Portfolio.
- There are important caveats regarding the PDI:
  - When used for portfolio construction, equal-weighted portfolios will not reflect the Naïve Risk Parity characteristics being measured by the PDI.
  - The PDI is unable to distinguish between correlations of different signs, and therefore can generate identical values for very different sets of correlations.


Fragkiskos, A. “What is Portfolio Diversification?” State Street White Paper (September 2013).


