Backtesting and Estimation Error: Value-at-Risk Overviolation Rate*

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Value-at-Risk (VaR)

- **The Question Being Asked in VaR:** "What severity of loss can be expected no more than x% of the time?
- VaR captures an important aspect of risk (how bad can it get?) in a single number and is easy to understand
- VaR has become the standard for market risk measurement in financial institutions
- Regulators base the capital they require banks to keep on VaR

VaR in the News: SEC Rule 18F-4

- Requires calculation and reporting of VaR for all investment funds that make substantial use of financial derivatives
- Losses in excess of threshold* must be report to SEC
- Backtesting exceptions (loss frequencies in excess of estimated VaR) must be reported to Board of Directors
- Implementation date Aug 19 2022

*VaR must not exceed 200% of the VaR of its designated reference portfolio or 20% of the value of the fund's net assets

How do we do this?

- 18f-4 serves a compliance objective
- Whatever methods you use for compliance, you’ve got to be able to explain clearly!
  - Regulators hate the idea of people using methods they don’t understand… and you’d better not use procedures that are prone to calculation errors!
- Nonetheless, methods must be good enough to stand up to backtesting.
  - Use of fudge factors to produce a more conservative result is not encouraged...

Basic Implementation of Parametric VaR

- Calculate variance/covariance matrix from historical series of daily returns of all constituents of the portfolio

$$\widehat{\sigma^2} = w'\hat{S}w$$

Where w are the current portfolio weights, and $\hat{S}$ = the estimated variance/covariance matrix of asset returns

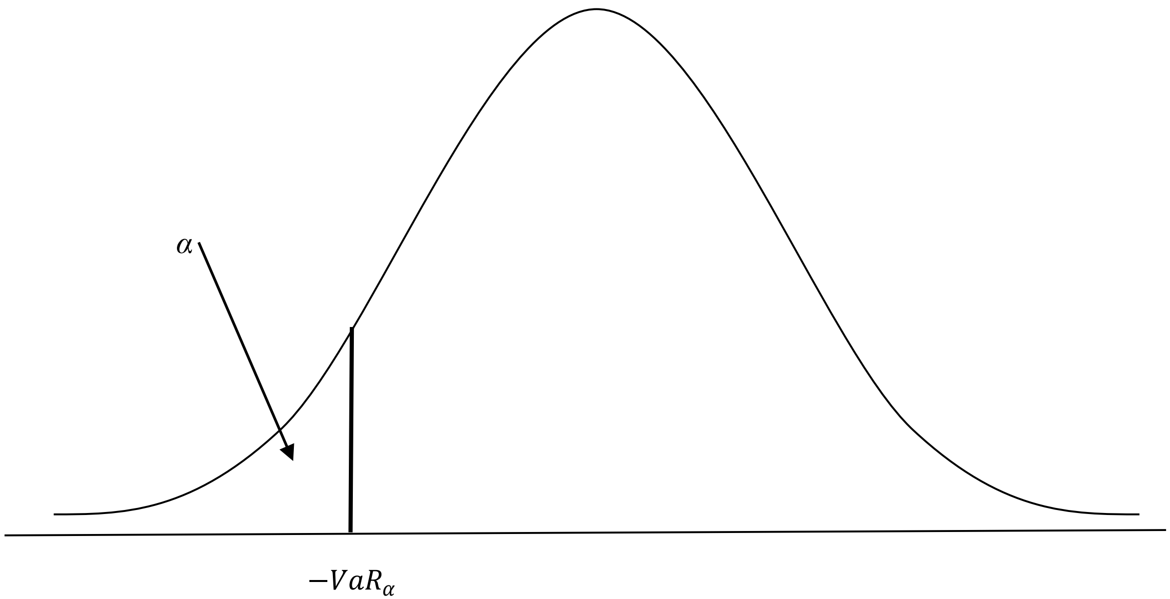
- Calculate VaR at the desired risk threshold level using desired probability distribution
- Keep track of actual vs predicted number of loss events in excess of threshold (backtesting)

Background to our Work

- Being wrong has serious consequences!
- Most research focuses on *"How do we improve the estimates of standard error"*?, Or *"What is an appropriate distribution for the returns process"*?
- Our approach recognizes estimation error as intrinsic to VaR estimation, and assesses its practical impact.
- We find that the estimation error systematically leads to ***overviolation*** of VaR thresholds under quite general conditions
- We develop practical methods to address the issue, grounded in basic statistics

What is the source of inaccuracy?

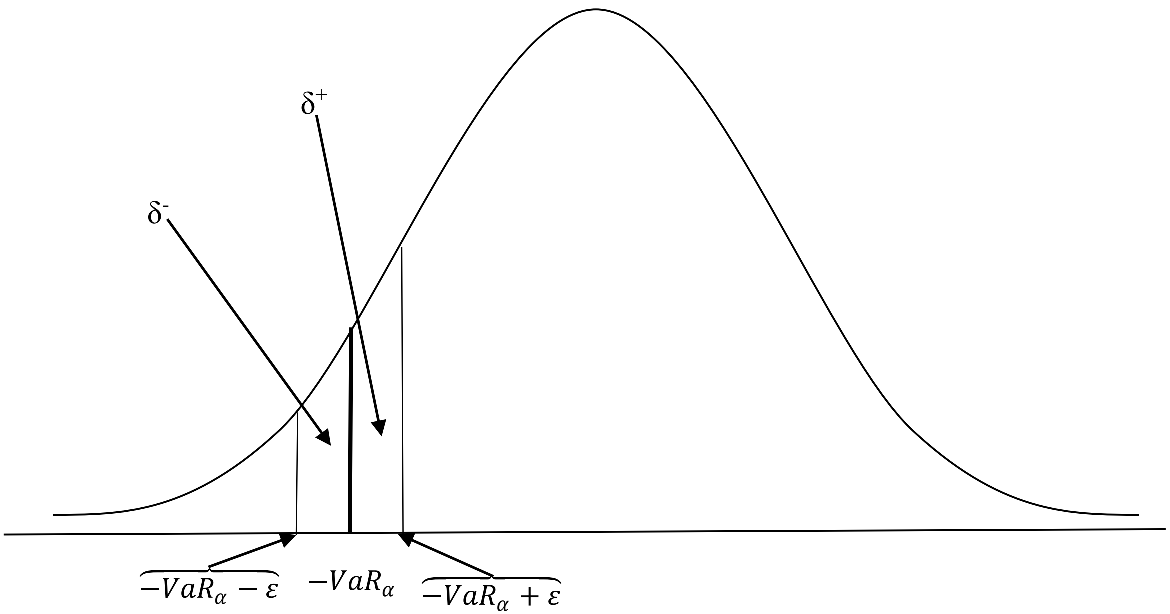
- Given the gain/loss distribution, the VaR can be defined as a quantile



- The VaR_α is the loss level such that the probability to exceed that loss is α

VaR Overviolation Illustration

- Suppose an error in the estimation of the VaR by $\pm\varepsilon$ as illustrated below



- The key point here is that $\delta^+ > \delta^-$: when we move to the right, the area increases more than it decreases when we move to the left

VaR Overviolation Illustration

- Consider an estimator of the VaR_α with the following distribution

$$\hat{v} = \begin{cases} VaR_\alpha - \varepsilon, & with\ Probability\ \frac{1}{4} \\ VaR_\alpha, & with\ Probability\ \frac{1}{2} \\ VaR_\alpha + \varepsilon, & with\ Probability\ \frac{1}{4} \end{cases}$$

However, by denoting $\hat{\alpha} \equiv \Pr(X < \hat{v})$, the violation rate from the estimate, we have that

$$E(\hat{\alpha}) = E(\Pr(X < \hat{v})) = \alpha + \frac{\delta^+ - \delta^-}{4} > \alpha.$$

- The average of violation rates is larger than the expected rate

Framework

- In the parametric framework assuming a Normal distribution with zero mean, the VaR can be written as follows:

$$VaR_\alpha = \sigma_t \Phi^{-1}(1-\alpha)$$

  where Φ is the cdf of the standard Normal distribution
- The VaR estimation will only need the volatility estimation

A common practice is to use the sample estimate

$$\hat{\sigma}_t = \sqrt{\frac{\sum_{\tau=t-m}^{t-1}(X_\tau - \bar{X})^2}{m-1}}$$

where $\bar{X}$ is the sample mean and m the estimation window

Framework

- Violation: For each period t, the actual return X_t can be compared to the value-at-risk forecast, $VaR_{\alpha,t}$
- Let denoting $I_t = \begin{cases} 1 \;\; if \;\; -X_t > VaR_{\alpha,t} \\ 0 \;\; if \;\; -X_t \leq VaR_{\alpha,t} \end{cases}$

Proposition 1. *In the Normal setup, the expected theoretical rate of violation*

$$E\left[\frac{1}{T}\sum_{t=1}^{T} I_t\right] \text{ is larger than } \alpha$$

Proposition 2. *This result is also true for a general distribution as far as the probability density function is increasing around the VaR (in other words, as far as the cdf is convex around VaR)*

An Example of Overviolation

- Some evidence in the literature: Berkowitz and O'Brien (2002, Journal of Finance)

Bank P&L and VaR Summary Statistics, August 1998 to October 1998

This table reports daily profit and loss data as reported by large commercial banks in the wake of the Russian default crisis, August 1998 to October 1998. For further details on the data, see Table I.

		Daily P&L					Daily VaR		
	Obs	Mean	Standard Deviation	Minimum	Excess Kurtosis	Skew	Mean VaR	Number Violations	Mean Violation
Bank 1	63	0.175	1.76	−7.01	4.58	−1.32	−2.32	3	−2.12
Bank 2	64	0.076	1.89	−4.26	1.46	0.787	−2.28	5	−0.862
Bank 3	65	−0.907	1.84	−8.68	7.65	−2.53	−4.63	3	−3.18
Bank 4	63	0.0453	0.773	−1.89	0.99	−0.434	−4.66	0	
Bank 5	65	0.0637	1.60	−5.51	1.99	−0.837	−5.09	1	−0.775
Bank 6	65	0.171	2.28	−14.2	41.1	−5.92	−1.42	2	−7.99

- The VaRs are at 1%, we should therefore expect (less than) 1 violation

Applying our framework

- A practical approach to adjustments:
  - The overviolation can be interpreted as a risk of systematic underestimation. Therefore scale up the VaR forecast can better account for as follows:

$$VaR_{\alpha}^{*} = (1 + \lambda^{*})VaR_{\alpha}$$

where λ^{*} is the solution to the equation

$$E\left[\Phi\left(-\frac{(1+\lambda)\sigma_t}{\sigma}\Phi^{-1}(1-\alpha)\right)\right] = \alpha$$

Computation of the adjustment

- Closed form solution infeasible
- Can be done directly through Monte Carlo Simulation or finding solutions by quadrature technique
- We use the quadrature technique
  - Involves discretizing the integral
  - Monte Carlo is used later to assess the overviolation and the adjustment

Monte Carlo Study

Setup

- We simulate 10,000 returns from a standard Normal distribution. The variances are generated with following alternative setups:
    i. Simulated variances from a Chi-square distribution
    ii. MA: $\hat{\sigma}_t = \sqrt{\frac{\sum_{\tau=t-m}^{t-1}(X_\tau - \bar{X})^2}{m-1}}$
    iii. EWMA: $\sigma_t^2 = 0.94\sigma_{t-1}^2 + 0.06X_{t-1}^2$
    iv. GARCH (1,1): $\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha X_{t-1}^2$
- We perform 1,000 runs and take the average, median and standard error

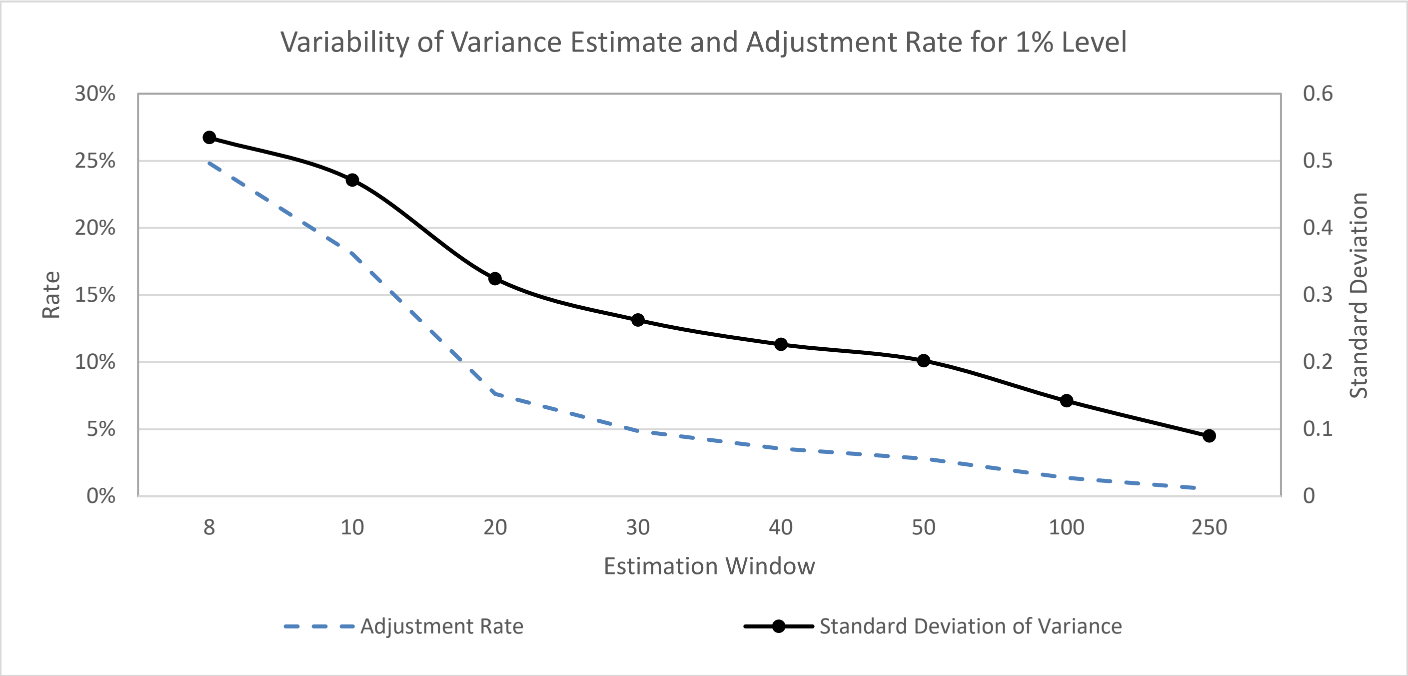
Monte Carlo Study

Estimation Window

- We use rolling window for MA estimation
  - Note that EWMA and GARCH are also forms of “rolling window” with effective lookback period determined by the decay factor in EWMA, and internally estimated in GARCH
- We analyze different window widths
  - For MA estimation, small window puts more weight on the recent information
  - In practice, the limited available data also leads to the use of small windows
  - Stress-testing by the Fed provides quarterly data which reduce the data frequency

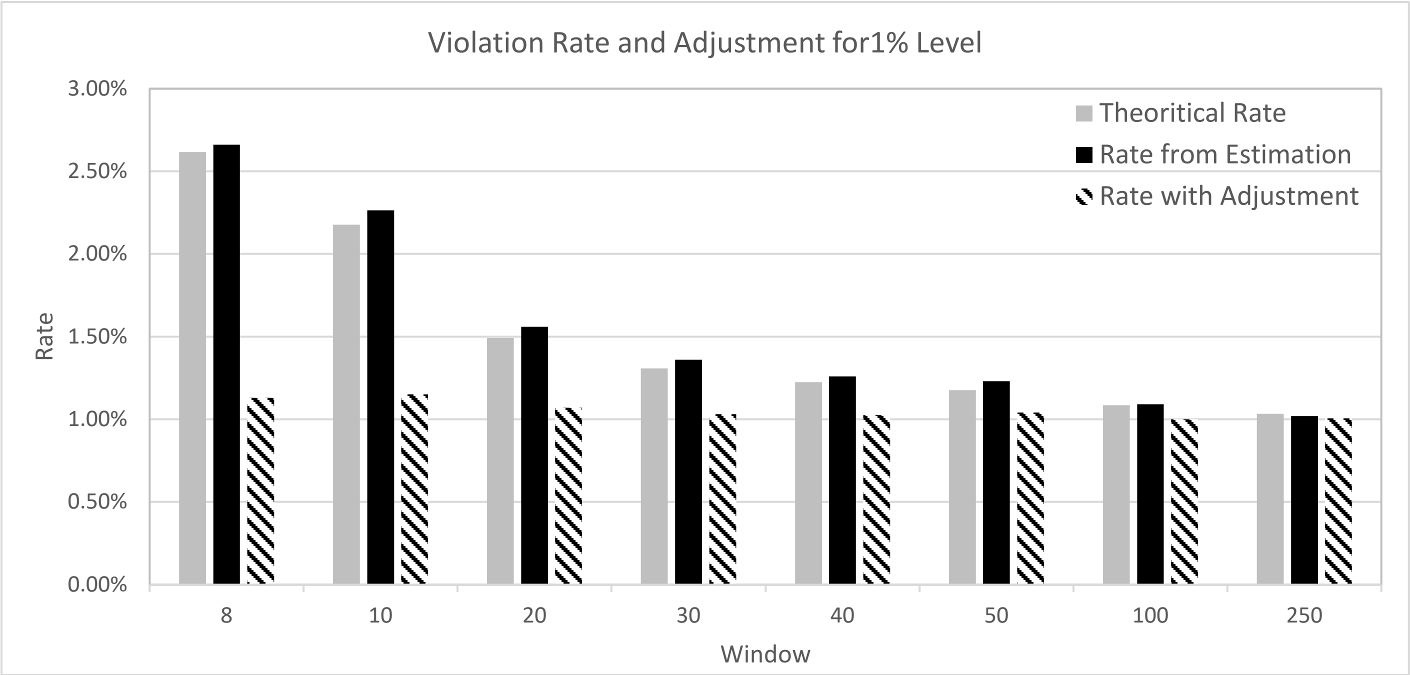
Monte Carlo Study

- When the estimation window is small, the variability of the variance estimate in MA is large
- The adjustment rate follows the same trend as the variance variability and tends to vanish for large window



Monte Carlo Study

- Overviolation is also inversely related to the estimation window.
- Theoretical rate and rate from MA estimation are very close.
- Effectiveness: The adjustment pushes the violation rate at its expected level (1% here)



Monte Carlo Study

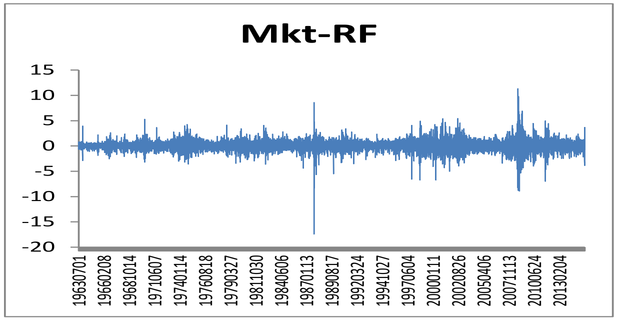
- Simulation Results for 0.5% VaR
  - Theoretical rates are close to both rates obtained with Simulated variance and MA estimation
  - The adjustment pushes the rate closer to 0.5%
  - Over-violation is less severe for EWMA and GARCH

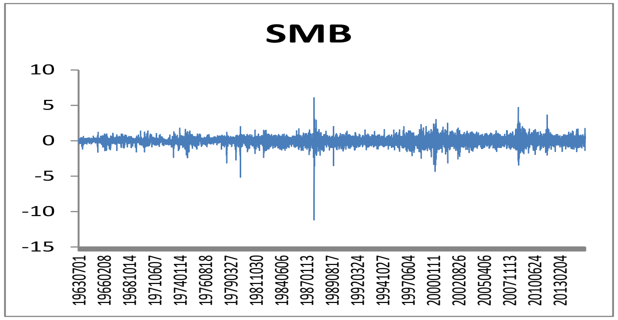
window (m)		Adjustment (λ^*)	Theo. Rate	Simulate	MA	Adj. MA	EWMA	GARCH
8	Median	**28.2601**	1.7279	1.8250	1.8500	0.6350	0.7050	0.6200
	Mean	28.2601	1.7279	1.8295	1.8431	0.6411	0.7197	0.6246
	St_dev	0.0000	0.0000	0.1167	0.1366	0.0746	0.0862	0.0788
10	Median	20.4914	1.3719	1.4800	1.4850	0.6100	0.7200	0.6200
	Mean	20.4914	1.3719	1.4874	1.4838	0.6235	0.7154	0.6172
	St_dev	0.0000	0.0000	0.1027	0.0928	0.0709	0.0782	0.0762
…	…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…	…
100	Median	1.4472	0.5575	0.5700	0.5700	0.5200	0.7300	0.6300
	Mean	1.4472	0.5575	0.5703	0.5721	0.5175	0.7208	0.6256
	St_dev	0.0000	0.0000	0.0744	0.0717	0.0731	0.0835	0.0793
250	Median	0.5856	0.5224	0.5250	0.5250	0.5000	0.7200	0.6250
	Mean	0.5856	0.5224	0.5310	0.5307	0.5050	0.7305	0.6301
	St_dev	0.0000	0.0000	0.0773	0.0708	0.0736	0.0778	0.0749

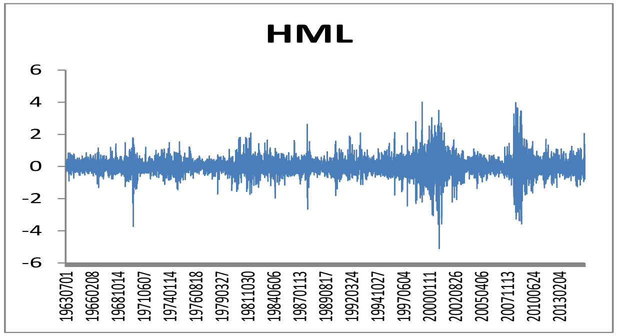
- Results pattern are very similar for 1% and 5% VaR

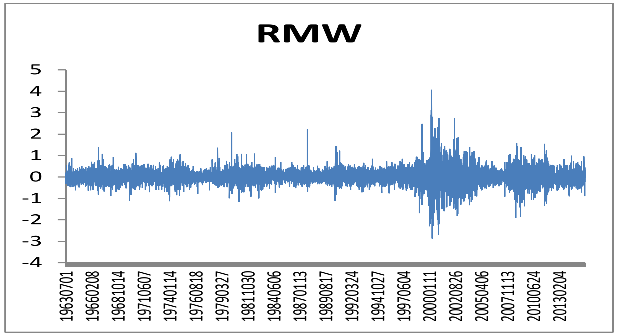
Application to Market Data

- We use four Fama-French portfolio returns: market excess return (Mkt – RF); size portfolio return (SMB); book-to-market ratio return (HML); and momentum portfolio return (RMW)
- Daily returns from July 1, 1963 to August 31, 2015, for 13,132 total observations.









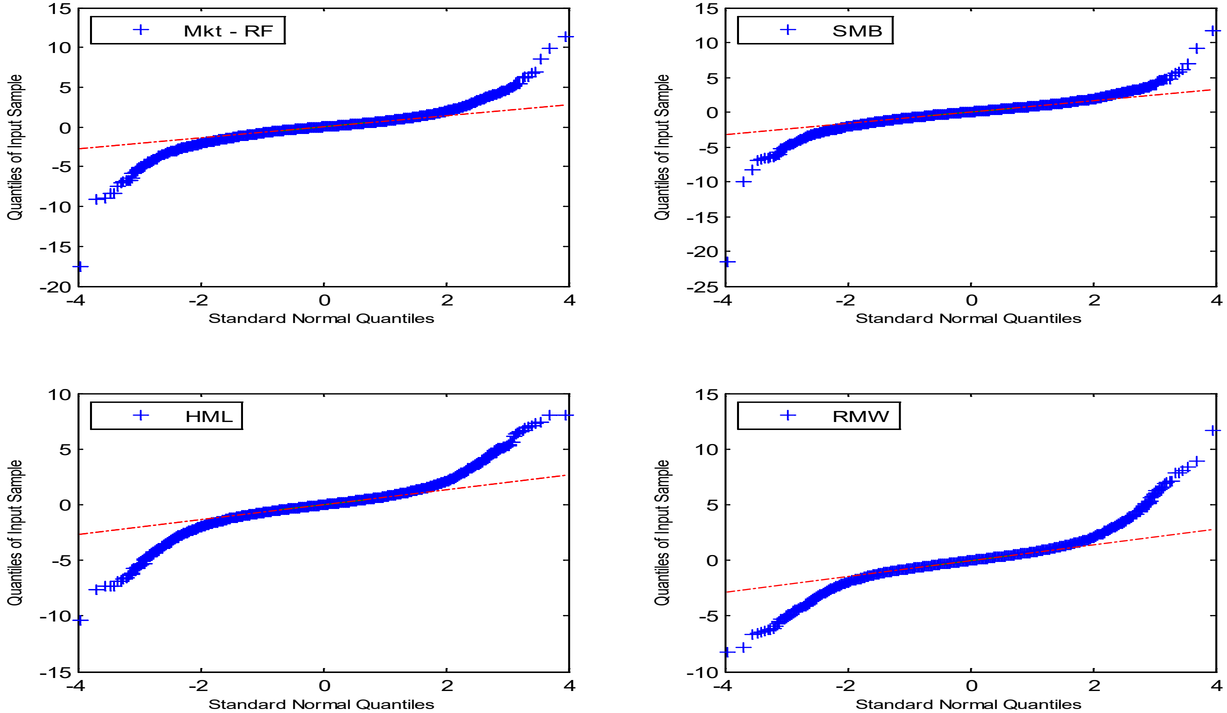
Application to Actual Data

- Descriptive Statistics
  - The main point here is the kurtosis. For all series, it is larger than 3, suggesting the presence of fat-tail

	Mkt-RF	SMB	HML	RMW
Mean	0.0238	0.0091	0.0170	0.0126
Standard Error	0.0086	0.0045	0.0043	0.0030
Median	0.0500	0.0300	0.0100	0.0100
Standard Deviation	0.9911	0.5201	0.4946	0.3474
Sample Variance	0.9823	0.2705	0.2447	0.1207
Kurtosis	**15.8624**	**22.4895**	**8.7904**	**9.5402**
Skewness	-0.5102	-1.0081	0.0845	0.4048
Minimum	-17.4400	-11.1900	-5.1200	-2.8600
Maximum	11.3500	6.1000	4.0100	4.0600
Number of Obs.	13132	13132	13132	13132

Data Distribution

- The QQ-Plots and statistical tests strongly reject the Normal Distribution for all four series



	Mkt-RF	SMB	HML	RMW
Jarque-Bera Statistic	138130	278750	42259	50117
(p-value)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Kolmogorov-Smirnov Statistic	0.4820	0.4908	0.4888	0.4918
(p-value)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Empirical Results

- At 0.5% VaR level, the Normal distribution fails to provide good forecasts
- Normal distribution is unable to capture fail-tails in data
- The adjustment cannot provide enough buffer
- Similar pattern for other Series

			Mkt-RF				SMB			
Window	**Adjustment**	**Theo. Rate**	**MA**	**Adj. MA**	**EWMA**	**GARCH**	**MA**	**Adj. MA**	**EWMA**	**GARCH**
8	28.2601	1.7279	2.5115	1.0506	1.2908	1.1007	2.4215	0.8205	0.9706	0.7805
			(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0011)
			[0.0000]				[0.0000]			
10	20.4914	1.3719	2.2418	1.1709	1.2910	1.1009	2.0216	0.9307	0.9708	0.7806
			(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0010)
			[0.0000]				[0.0000]			
…	…	…	…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…	…	…	…
100	1.4472	0.5575	1.3836	1.2826	1.3028	1.1210	1.1412	1.0301	0.9695	0.7776
			(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0013)
			[0.0000]				[0.0000]			
250	0.5856	0.5224	1.2818	1.2715	1.2818	1.1280	1.2818	1.2613	0.9844	0.7793
			(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0013)
			[0.0000]				[0.0000]			

Empirical Results

- At 1% level, the Normal distribution improves forecasts slightly
- The adjustment also starts to be effective

			HML				RMW			
Window	Adjustment	Theo. Rate	MA	Adj. MA	EWMA	GARCH	MA	Adj. MA	EWMA	GARCH
8	24.8087	2.6160	3.1919	1.4008	1.2508	1.1607	3.1819	1.4509	1.5309	1.3308
			(0.0000)	(0.0001)	(0.0044)	(0.0032)	(0.0000)	(0.0000)	(0.0000)	(0.0001)
			[0.0003]				[0.0004]			
10	18.0813	2.1772	2.6321	1.3811	1.2510	1.1609	2.6721	1.4912	1.5312	1.3311
			(0.0000)	(0.0001)	(0.0044)	(0.0032)	(0.0000)	(0.0000)	(0.0000)	(0.0001)
			[0.0018]				[0.0007]			
20	7.6294	1.4910	1.7331	1.3625	1.2523	1.1521	2.0136	1.4526	1.5328	1.3324
			(0.0000)	(0.0003)	(0.0043)	(0.0033)	(0.0000)	(0.0000)	(0.0000)	(0.0001)
			[0.0459]				[0.0000]			
30	4.8554	1.3086	1.4440	1.2335	1.2435	1.1532	1.8652	1.4541	1.5343	1.3337
			(0.0000)	(0.0191)	(0.0050)	(0.0033)	(0.0000)	(0.0000)	(0.0000)	(0.0001)
			[0.2340]				[0.0000]			
40	3.5585	1.2248	1.4053	1.1845	1.2447	1.1544	1.7366	1.5158	1.5358	1.3351
			(0.0000)	(0.0642)	(0.0049)	(0.0033)	(0.0000)	(0.0000)	(0.0000)	(0.0001)
			[0.1013]				[0.0000]			
50	2.8079	1.1767	1.2761	1.1455	1.2460	1.1555	1.7283	1.5977	1.5374	1.3364
			(0.0008)	(0.1446)	(0.0048)	(0.0032)	(0.0000)	(0.0000)	(0.0000)	(0.0001)
			[0.3578]				[0.0000]			
100	1.3662	1.0854	1.3735	1.2523	1.2523	1.1614	1.6966	1.5552	1.5249	1.3432
			(0.0000)	(0.0116)	(0.0043)	(0.0030)	(0.0000)	(0.0000)	(0.0000)	(0.0001)
			[0.0057]				[0.0000]			
250	0.5301	1.0334	1.3023	1.2715	1.2510	1.1587	1.5279	1.4869	1.5279	1.3638
			(0.0000)	(0.0070)	(0.0043)	(0.0029)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
	24.8087	2.6160	3.1919	1.4008	1.2508	1.1607	3.1819	1.4509	1.5309	1.3308

Empirical Results

- At 5%, the Normal distribution provides good forecasts
- The adjustment is mostly effective here

			HML				RMW			
Window	Adjustment	Theo. Rate	MA	Adj. MA	EWMA	GARCH	MA	Adj. MA	EWMA	GARCH
8	16.1022	7.4372	7.8547	5.4933	5.2031	4.9730	8.2650	5.7835	5.3132	4.9230
			(0.0000)	(0.0237)	(0.0294)	(0.0181)	(0.0000)	(0.0003)	(0.0005)	(0.0006)
			[0.1116]				[0.0016]			
10	11.9954	6.8682	7.3759	5.4844	5.1841	4.9740	7.3859	5.6545	5.3143	4.9139
			(0.0000)	(0.0263)	(0.0284)	(0.0182)	(0.0000)	(0.0027)	(0.0005)	(0.0005)
			[0.0448]				[0.0407]			
20	5.2721	5.8614	6.1411	5.1493	5.2094	4.9689	6.3815	5.5300	5.3196	4.9189
			(0.0000)	(0.4938)	(0.0016)	(0.0173)	(0.0000)	(0.0151)	(0.0005)	(0.0005)
			[0.2343]				[0.0270]			
30	3.3795	5.5596	5.5355	4.9840	5.1845	4.9639	5.7962	5.2948	5.3249	4.9238
			(0.0000)	(0.9414)	(0.0276)	(0.0164)	(0.0000)	(0.1767)	(0.0005)	(0.0005)
			[0.9163]				[0.3025]			
40	2.4867	5.4144	5.3102	4.9588	5.1696	4.9689	5.4909	5.2299	5.3303	4.9388
			(0.0009)	(0.8505)	(0.0263)	(0.0167)	(0.0000)	(0.2925)	(0.0005)	(0.0006)
			[0.6457]				[0.7360]			
50	1.9670	5.3290	5.1346	4.8432	5.1748	4.9739	5.4160	5.1547	5.3356	4.9337
			(0.0002)	(0.4731)	(0.0263)	(0.0170)	(0.0000)	(0.4788)	(0.0005)	(0.0006)
			[0.3880]				[0.6993]			
100	0.9619	5.1621	4.9687	4.8172	5.1808	4.9586	5.2616	5.1808	5.3121	4.9182
			(0.0000)	(0.4040)	(0.0247)	(0.0140)	(0.0000)	(0.4092)	(0.0004)	(0.0004)
			[0.3845]				[0.6545]			
250	0.3778	5.0642	4.8913	4.8503	5.1579	4.9938	5.1989	5.1887	5.3015	4.8913
			(0.0000)	(0.4975)	(0.0297)	(0.0236)	(0.0000)	(0.3926)	(0.0003)	(0.0005)
			[0.4360]				[0.5441]			

Refining the Method

- The inability of adjustment derived from Normal distribution to adjust sufficiently for more prudential levels (0.5% and 1%) is due to the fat-tails in the data
- We replace the Normal distribution by the empirical distribution
- This is a semi-parametric approach also called “hybrid” method by practitioners
- How does this perform?

Empirical Results

For the market portfolio:

- Conservative at 0.5%
- Accurate at 1%
- A little bit off at 5%

	Mkt-RF											
		0.5%				1%				5%		
Window	Theo. Rate	MA	Adj. MA		Theo. Rate	MA	Adj. MA		Theo. Rate	MA	Adj. MA	
8	1.7279	0.7705	0.2401		2.6160	1.7511	0.8305		7.4372	8.3750	6.1337	
		(0.0016)	(0.0002)			(0.0000)	(0.0886)			(0.0000)	(0.0000)	
		[0.0000]				[0.0000]				[0.0004]		
10	1.3719	0.5604	0.2902		2.1772	1.4912	0.7306		6.8682	7.8763	6.0649	
		(0.0882)	(0.0030)			(0.0000)	(0.0068)			(0.0000)	(0.0000)	
		[0.0000]				[0.0000]				[0.0001]		
20	0.8452	0.3106	0.1903		1.4910	1.0218	0.7514		5.8614	7.0327	6.2913	
		(0.0039)	(0.0000)			(0.0029)	(0.0125)			(0.0000)	(0.0000)	
		[0.0000]				[0.0001]				[0.0000]		
...	...	...	...	...	...	...	...	...	...	...	...	
50	0.6204	0.2613	0.2311		1.1767	0.8039	0.6933		5.3290	6.4811	6.1596	
		(0.0002)	(0.0001)			(0.0003)	(0.0021)			(0.0000)	(0.0000)	
		[0.0000]				[0.0006]				[0.0000]		
100	0.5575	0.2828	0.2828		1.0854	0.8382	0.7978		5.1621	6.0594	5.9887	
		(0.0007)	(0.0022)			(0.0000)	(0.0432)			(0.0000)	(0.0000)	
		[0.0002]				[0.0176]				[0.0001]		
250	0.5224	0.3589	0.3384		1.0334	0.8511	0.8101		5.0642	5.9783	5.9270	
		(0.0000)	(0.0237)			(0.0001)	(0.0595)			(0.0000)	(0.0000)	
		[0.0251]				[0.0750]				[0.0000]		

Conclusion

- We shed light on an important issue of overviolation even in the case of an unbiased estimator
- We provide some appropriate adjustments and show their effectiveness in a Monte Carlo study.
- When applying in the data, we suggest an "hybrid" approach (combining volatility estimation with empirical distribution) to account for tail behavior
- Future related extension can involve analyzing the issue in a multivariate setup with correlation estimation or more general dependence structure.

Conclusion

Thank You!