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Risk and Optimization with Uncertain Portfolio Weights

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Motivation

Within the calculations required for portfolio risk analyses and optimization, it is always true that we incorporate the relevant information about the expected variability of the returns of the assets.

The other ingredient to such calculations is the asset weight vector of the current portfolio state, *the elements of which are considered known with certainty* at each moment in time. This formulation is entirely consistent classical portfolio theory (Markowitz 1952) that is a *single period* model where the costs of rebalancing portfolio weights from time to time are ignored.

- *Under the **false** assumption that transaction costs are zero, we can justify the assumption that investors rebalance their portfolios in continuous time to keep asset weights fixed.*

However, if we assume transaction costs are non-zero, optimal investment management requires that portfolio weights be revised only from time to time.

Today's Outline

Between rebalancing events asset weights will move with future market prices, thereby introducing uncertainty into the values of asset weights that are the most representative input for the time interval between rebalancing events.

We will show that most of the "uncertainty" around portfolio weights is intuitive in how it biases our understanding of portfolio risk and optimal weights.

In this presentation, we will describe an analytical solution to how portfolio risk assessments and portfolio optimization procedures can be improved to address this issue.

In addition, we will describe how this issue can be mitigated by more thoughtful management of investment cash flows (e.g. dividends, bond coupon payments) to minimize asset weight uncertainty and thereby reduce the transaction and tax costs of periodic portfolio rebalancing.

Simple Statements

- The traditional representation of portfolio risk is:

$$\Sigma_{i=1 \text{ to } n} w_i \, \Sigma_{j=1 \text{ to } n} w_j \, (\sigma_i \sigma_j p_{ij})$$

- If we want to be extra careful, we should express the covariance term as an expectation since it is descriptive of future outcomes, not the current state of the portfolio

$$\Sigma_{i=1 \text{ to } n} w_i \, \Sigma_{j=1 \text{ to } n} w_j \, E[(\sigma_i \sigma_j p_{ij})]$$

- There is a vast literature in how the expectation of the covariance matrix can be in error, and the methods to mitigate that error. See Northfield Tools for Addressing Estimation Error in Portfolio Construction (northinfo.com).

- *But we routinely take the portfolio weights as **known values**, which is only true instantaneously*

Optimal Portfolio Rebalancing

If the assumption of continuous rebalancing to keep portfolio weights fixed is unrealistic:

- We must admit that between now and whenever the portfolio is to be rebalanced the next time, the portfolio weights are themselves going to fluctuate with market prices.
- The way in which portfolio income (dividends and interest) is reinvested will also impact weights

To the extent that the "single period" representation no longer applies, the Northfield optimizer adjusts the process for an attribute that we call "probability of realization" as a way to approximate correct multi-period tradeoffs

- Again, there is an extensive literature on how often to rebalance portfolios back to optimal weights, and various procedures to properly consider both the benefits (higher portfolio utility) and the costs.
- See Smarter Rebalancing: Using Single Period Optimization In a Multi-period World (northinfo.com)

A Basic Intuition

Between rebalancing events:

- Each element of the portfolio weight vector will move as dictated by price movements.
- The portfolio weight vector must still sum to 100%

Assets with higher expected returns are more likely to experience relative increases in weight.

- Portfolio assets with lower expected returns are more likely to experience relative decreases in weight.

Assets with higher magnitude of idiosyncratic risk may be expected to have their weights randomly move around more than assets with low magnitude of idiosyncratic risk.

- In effect, we have to think about idiosyncratic risk as having an exaggerated effect as it both reflects the returns the portfolio will experience, but also degree of estimation error arising from using scalar weight values.
- This effect is often most noticeable in market neutral portfolios where idiosyncratic risk usually dominates factor risks.

Remaining Effects

Assets that have CAPM beta close to one *relative to the rest of the portfolio* are less like to experience weight changes.	• Assets with CAPM beta away from one will have more weight movement
Assets with CAPM beta > 1 will be expected to have weight increases in an up market and weight decreases in a market decline.	• Up markets increase the absolute risk of the portfolio (riskier assets become more predominant)
Assets with CAPM beta < 1 will be expected to have weight decreases in an up market and increases in a market decline.	• Down markets decrease the absolute risk of the portfolio (riskier assets become less predominant).
Between rebalancing events, we can treat the elements of the portfolio weight vector as random variables each with an expected value and standard deviation over the course of the time interval.	• If the number of portfolio assets is relatively large, the constraint that weights sum to 100% has minimally binding effect.

What's Old is New Again

What is Fuzzy?

- Fuzzy means
- not clear, distinct or precise;
- not crisp (well defined);
- blurred (with unclear outline).

The idea of treating portfolio weights as random variables has been around for quite a while.

- Nearly twenty years ago, Northfield's Rick Gold published a paper called "Why the Efficient Frontier for Real Estate is Fuzzy" (*Journal of Real Estate Portfolio Management*, 1995).
- This paper argues that for illiquid assets investors must think of both current and optimal portfolio weights must be thought of as distributions rather than point values, the concept of keeping weights fixed through even relatively infrequent rebalancing is impractical.

The paper uses bootstrap resampling of the optimization input parameters to estimate the magnitude of the confidence interval on each portfolio weight.

In addition, some illiquid assets such as "brick and mortar" real estate are largely indivisible, where a holding cannot be sold in part, but only kept or sold in full.

- Optimizing such portfolios requires special techniques
  - See Belev and Gold (*American Real Estate Society* Practitioner Paper of the Year, 2015)

Risk Decomposition By Asset

- Expected portfolio risk is often expressed in units of standard deviation or scalars of standard deviation (e.g. Value at Risk).
  - For non-normal distributions, see diBartolomeo (*Journal of Performance Measurement*, 2023)

- Underlying calculations are routinely done in variance units which are naturally additive across both portfolio positions and across time intervals.

- If we wish to attribute portfolio variance out to the positions in the portfolio, this is easily done.

$$\text{Variance Contribution}_{(i)} = d(\text{portfolio variance})/d(\text{weight of asset}_{(i)}) * .5 * (W_i - B_i)$$

- For tracking variance, both the weights in the portfolio and the weights in a traditional benchmark will be random variables.
  - For absolute return variance the benchmark can be fixed as 100% cash

Putting It All Together

- For real world problems, both asset and benchmark weights are more correctly represented as random variables rather than known scalar values.
  - The longer the intervals between rebalancing events, the greater the magnitude of the variability of portfolio weights.
  - We should recognize both the covariance matrix and the portfolio weights as expectations

$$E[\Sigma_{i=1 \text{ to } n} w_i \Sigma_{j=1 \text{ to } n} w_j] \; E[(\sigma_i \sigma_j p_{ij})]$$

- Variations in the portfolio weights arise as combination of the expected returns, the covariance matrix, and the estimation errors of the covariance matrix.

- A very complex, but closed form solution to the combined problem is presented in Shah (2023) [Portfolio Risks under Estimation Uncertainty and Price Movement by Anish Shah :: SSRN]

A Cheaper Solution

One obvious, but rarely used method to mitigate drifting of portfolio weights is to direct reinvestment portfolio income (dividends and interest) to offset the drifting of portfolio weights when possible.

Such "pro-active" management of the asset weight vector reduces the fluctuations in portfolio weights and often the lengthens the optimal period between rebalancing events.

Reducing the frequency and required magnitude of rebalancing trades can have a material effect on both realized transaction costs and capital gain taxes (if applicable).

Two papers by Balvers and Mitchell (1997 and 2000) show that what they call "efficient gradualism" has economically material benefits.

Conclusions

While it is customary for investors to formulate portfolio risk and optimization problems as "single period", this is just an approximation of the real world where asset weights are themselves random variables during the time interval between now and the next rebalancing event.

Just as practice has advanced to routinely consider estimation error in asset return covariance matrices, we should understand how scalar values for portfolio weights *can bias both risk forecasts and optimal weight calculations in intuitive and predictable ways.*

Although complex, analytical solutions exist for the "full uncertainty" problem where there is estimation error in the covariance matrix, and uncertainty in the portfolio weights due to finite rebalancing frequency.