Even trained statisticians often fail to appreciate the extent to which statistics are vitiated by the unrecorded assumptions of their interpreters.... It is easy to prove that the wearing of tall hats and the carrying of umbrellas enlarges the chest, prolongs life and confers comparative immunity from disease. A university degree, a daily bath, the owning of thirty pairs of trousers, a knowledge of Wagner's music, a pew in church, anything, in short, that implies more means and better nurture can be statistically palmed off as a magic spell conferring all sorts of privileges. The mathematician whose correlations would fill Newton with admiration, may, in collecting and accepting data and drawing conclusions from them fall into quite crude errors by just such popular oversights as I have been describing.

George Bernard Shaw
_The Doctor’s Dilemma_
Introduction
When we consider the risk of investing in equity securities, we really face three separate problems. The first is to come to a definition of “risk” that is an appropriate representation of our preferences among various possible performance outcomes. Such a framework for risk must also be intuitively appealing and understandable. It must be mathematically tractable enough to be made a routine part of our portfolio management process.

Our second task is to develop methodologies for measuring and forecasting the risk of equity securities and portfolios. Within current investment practice, a wide variety of models of equity risk exist, each with its particular strengths and weaknesses. Several of these methodologies will be reviewed.

Finally, we will take up the issue of estimation error. Whatever our definition of risk and however carefully we estimate the future risk of our equity investments, we must always be concerned about the possibility that our forecast is simply wrong. Most implementations of portfolio theory assume that both our definition of risk and our forecasts of those risks are perfect. Almost universally, we ignore the ‘risk’ that our understanding of risk is itself flawed.

The Framework for Risk in Equities
When we talk about risk, we must first make a distinction between the uncertainty of a future outcome (statistical risk) and the potential for undesirable outcomes (economic risk). Imagine receiving a gift of a lottery ticket, with a condition from the donor that we not resell the ticket. In accepting such a gift and awaiting the outcome of the lottery, we endure no economic risk (the ticket cost nothing and the prohibition on resale removes opportunity costs), yet the uncertainty of the outcome is enormous. On the other hand, consider the act of suicide by jumping off a very tall building. The uncertainty of outcome is very small, but the risk of undesirable outcomes as measured by injury or death is tragically large.

A less abstract argument about the definition of risk is that investors do not have any objection to receiving surprisingly good returns. Hence, measures of dispersion such as standard deviation or variance are not appropriate to economic risks unless the probability distribution of returns is symmetric about the investor’s expectations. However, much study has been applied to the stock returns with little evidence that returns are other than symmetrically distributed about their means. Most research such as that of Sidney Alexander ("Price Movements in Speculative Markets: Trends or Random Walks", in Paul Cootner, ed., "The Random Character of Stock Market Prices", Cambridge, MA" MIT Press, 1964, Pg. 199-218) suggest that returns are not only symmetrically but normally distributed, strengthening the appropriateness of parametric dispersion measures.
As an alternative measure of risk, some investors choose to use methods based on the potential frequency and magnitude of returns below some target level. Such methods are referred to as “downside risk” or “lower partial moment”. For a good discussion see “Asset Allocation in a Downside Risk Framework” by W.V. Harlow, which appeared in Financial Analysts Journal in 1991. Such methods seem appropriate in two circumstances. The first is when the asset returns themselves have reliably forecastable skewness, as in options. The second is when the target level of return is far from the mean of the forecast asset returns. While such methods have achieved limited popularity in asset class allocation problems, the lack of asymmetries in stock returns have made this argument less compelling. In addition, most equity portfolio returns are measured relative to benchmarks which are specifically chosen because their expected return is near the center of the distribution of returns available from the selection universe of stocks. There are also some more recent academic studies, such as “Selecting Portfolios of Highly Skewed Assets Using Means, Variances and Higher Moments” by Hlawitschka and Stern (presented at Financial Management Association Annual Meeting, 1995), which suggest that lower partial moment methods are not beneficial to portfolio investors even when asset return distributions are skewed.

Assuming that we can accept a metric for risk, we must now consider the basis for comparison when we measure risk. In the usual institutional equity investment framework, we consider risk as the potential for differential (active) returns relative to the performance of some benchmark portfolio. There are two rationales for considering risk on a relative rather than absolute basis. The first is that investors usually make tradeoffs between forecasted returns and risks on an absolute basis during some asset allocation process. It is therefore desirable that the equity portfolio be similar in return behavior to the return behaviors that were forecast as a part of the asset allocation process. The second reason that relative risk is important is that institutional investors presume that there is persistence in relative manager performance. Therefore, they frequently use past performance as a criterion in selecting portfolio managers. By using a common benchmark index to “grade” the managers, it may be possible to compare more effectively portfolio managers of similar investment approaches. The issue of performance persistence has been the subject of many research studies such as “Hot Hands in Mutual Funds” by Hendricks, Patel and Zeckhauser (Journal of Finance, 1993) and “Performance Persistence” by Brown and Goetzmann (Journal of Finance, 1995). An exhaustive overview of the area may be found in Sneddon (“Does Performance Persist?”, Proceedings of 1996 Northfield Research Seminar). The risks to investors from improper classification of managers into peer groups is covered in Witkowski & diBartolomeo (”Mutual Fund Misclassification”, forthcoming in Financial Analyst Journal, 1997).

In contrast, there are many circumstances in which we may choose to measure risk in an absolute return/risk framework. For example, retail investors are generally more concerned with absolute losses of their capital than they are concerned with maintaining return performance that is superior to, or at least consistent with, a particular benchmark
Even in the institutional marketplace, it has been argued that investors’ utility functions really combine concerns about both relative risk and absolute risk. Discussion of this issue can be found in Wilcox (“Why EAFE is for Wimps”, Journal of Portfolio Management, 1994). It should be noted that absolute risk can be viewed as risk relative to a risk-free investment. As such, a joint benchmark consisting of some combination of the equity benchmark and the risk-free asset can be constructed so as to deal jointly with the dual risk preferences of some investors.

**Modeling Equity Risks**

If we accept that dispersion of potential outcomes as measured by the variance of future returns is an appropriate measure of risk, we can then set about the process of building models of equity risk. Such models can then be used to forecast the risk of individual stocks or portfolios of stocks.

It has long been intuitively apparent to investors that one way to control the risk of an investment portfolio is to diversify by holding portions of the portfolio in various different investments. To the extent that it was unlikely that all assets in a portfolio would coincidentally rise or fall in value at the same moment, the variance of portfolio returns would be reduced relative to a single large investment. Numerous academic papers such as “Diversification and the Reduction of Dispersion: An Empirical Analysis” by Evans and Archer (Journal of Finance, 1968) and “How Many Stocks Make a Diversified Portfolio?” by Statman (Journal of Financial and Quantitative Analysis, 1987) have made arguments as to construction of a diversified portfolio. It should be noted that almost all of these papers make the assumption that the stocks in a portfolio are selected at random. This assumption ought be carefully examined by active portfolio managers. To the extent that active portfolio management is by definition a process of selecting stocks that are somehow more attractive as investments than some other stocks (rather than picked at random), the diversification guidelines described in these papers cannot possibly apply. For example, if we believe that low price/earnings ratio stocks will outperform high price/earnings ratio stocks, we will choose to hold low price/earnings ratio stocks. Whether our portfolio consists of 10, 25 or 200 low price/earnings ratio stocks, the risk that our original premise, the expected outperformance of low price/earnings ratio stocks, may be wrong is not reduced or mitigated.

The beginnings of formal methods in investment risk management lie with the pioneering work of Markowitz (“Portfolio Selection”, Journal of Finance, 1952). With his mathematical construction of what came to be known as Modern Portfolio Theory, he showed how the variances of individual stock returns and the correlations of those returns can be combined to calculate a value for the variance of a portfolio made up of those stocks. In essence, Markowitz’s methods allow investors to diversify their portfolios in a purposeful rather than naïve fashion.
While Markowitz’s method is very popular for portfolio problems concerning asset classes, the traditional formulation is not widely used for equity portfolio problems. This is because it requires us to forecast not only the variance of return for each stock, but the correlation of return between each stock and every other stock involved in the problem. Imagine a portfolio measured against an S&P 500 benchmark. We would have to forecast 124,750 correlation coefficients. If we make the simplifying (and dangerous) assumption that the future will be like the past, we could use pairwise regression analysis to obtain statistical estimates of all past correlations. To the extent that these regression based values are themselves estimates, subject to sampling error, it is nearly inevitable that among our 124,750 correlation forecasts many will be proven inaccurate as future events unfold. One set of tests, in a paper by Elton, Gruber and Urich (“Are Beta’s Best?”, *Journal of Finance*, 1978), showed that in stock portfolio problems, it was effective to simply assume that all correlation coefficients were equal to the average of past correlation coefficients, rather than presume that each separate correlation coefficient would be equal to its individual past value.

The general solution to the shortcomings of the Markowitz method for estimate stock portfolio risk is the use of common factor models. In a common factor model, we establish a set of criteria by which we judge the level of similarity or difference between two or more portfolios of securities. If two portfolios have similar characteristics, we would expect their performance to be highly correlated. If their characteristics are dissimilar, we would expect their performance to have low or possibly negative correlation. To the extent that a single stock can be considered a portfolio with only one member, factor models can usually also be used to estimate the volatility of a specific stock. Other than the obvious fact that we should have grounds for believing that the criteria by which we judge similarity should have something to do with variations in stock returns, the number and nature of common factors we might choose to include in a model is subject to considerable debate.

Another way to think of this matter is that factor models provide a check on the reasonableness of the assumption that future correlations will be like past ones. If we had two securities that were highly correlated in the past and had very similar characteristics, we might think it reasonable that the highly correlated behavior would continue. However, if two stocks had very different characteristics, we might think that a high level of past correlation was a random coincidence. We can also make evaluations of the volatility of a given stock. If we note that a particular characteristic is associated with variations in stocks returns, securities having more extreme values of that characteristic would tend to be more volatile. In essence, we are able to separate what portions of risk arise from causes that are pervasive across many securities from risks that are genuinely specific to the stock of a particular company.
The most widely known common factor model is the single-index model. The Capital Asset Pricing Model developed by William Sharpe (“Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk”, *Journal of Finance*, 1964) is a special case of the single index model. In the usual implementation of the single index model, the common factor is the excess return over the risk free rate on a portfolio consisting of the entire equity market. Typically, time series regression analysis is used to estimate the relationship between the returns on a particular stock and the market return factor. The resultant measure of systematic (pervasive) risk is called “beta”. Variations in security returns which cannot be explained through beta are presumed to be stock specific, and hence uncorrelated between any pair of stocks. Several papers have been written such as those by Vasicek (“A Note on the Cross-Sectional Information in Bayesian Estimation of Security Betas”, *Journal of Finance*, 1973 and Blume (“Betas and Their Regression Tendencies”, *Journal of Finance*, 1975), that suggest methodological improvements over using simple regressions to estimate beta.

The CAPM is a special case of the index model, where we make the additional assumption that higher long term returns may be expected from stocks with higher levels of beta (more systematic risk). It should be noted that while there has been much controversy in recent years in the effectiveness of the CAPM (see Grinold, “Is Beta Dead Again”, *Financial Analysts Journal*, 1993) as a predictor of expected returns, little if any of the criticism of the CAPM has been directed to the use of beta as a measure of risk for well diversified portfolios. Many studies such as the “Conditional Relation Between Risk and Returns” by Petingill, Sundaram and Mathur (presented at Financial Management Association Annual Meeting, 1995) have confirmed the effectiveness of beta as a means of risk prediction.

Models with multiple common factors are currently the most popular mechanisms for predicting equity risk. The use of multiple factor models arises from the belief that while a single factor may describe a large portion of the common aspects of security returns, many other factors may influence some important subset of the universe of equities without having any influence on all securities. As an example, it might seem obvious that the variances and correlation of returns of two gold mining stocks would be influenced by the changes in the price of gold, as well as changes in broad economic conditions that are presumed to affect returns of equity securities in general.

Three types of multiple factor models are currently popular. The first is an exogenous factor model, where the common factors are typically economic state variables such as interest rates, levels of production, inflation, and energy costs. In essence, each stock is presumed to have several betas, each with respect to a particular aspect of the economy. If two securities (or portfolios) produce similar returns in response to shifts in the prescribed economic variables, they are presumed to be similar. Which aspects of the economy are important enough to include in such a model has been explored by Chen, Roll and Ross (“Economic Forces and the Stock Market, *Journal of Business*, 1986). Typically, such models include four to seven factors. While the changes in the economic variables may be
readily observed, the sensitivity of individual stocks to those changes must be statistically inferred. The beta values are usually estimated using time series regression analysis, as in the case of the single index model. The Arbitrage Pricing Theory (see Roll and Ross, “An Empirical Investigation of the Arbitrage Pricing Theory”, Journal of Finance, 1980) is the equilibrium form of this type of model. As in other factor models, security return variations not explainable through the common factor model are presumed to be stock specific and pairwise uncorrelated.

Proponents of specified macroeconomic factor models point out that such models typically exhibit stable behavior because they are tied to the real economy through genuinely pervasive factors. They also provide the opportunity for portfolio managers to gain a new level of insight into top-down economic effects on their portfolios and allow them to forecast likely performance under different scenario forecasts. The primary criticism of models with exogenously specified factors is that they cannot readily capture risks that are not part of the economic state. For example, such a model would not capture the product liability risks of tobacco companies.

The second type of model uses observable stock characteristics as proxies for factors of commonality. Such proxy factors might include stock fundamentals, like the price/earnings ratio, dividend yield, market capitalization, balance sheet leverage, and industry participation. For example, the relationship between leverage and risk was explored in “The Effect of the Firm’s Capital Structure on the Systematic Risk of Common Stocks” by Hamada (Journal of Finance, 1972). One might also add technical factors, such as price momentum and trading volume. This type of model was pioneered by Rosenberg (“Extra Market Components of Covariance in Security Returns”, Journal of Financial and Quantitative Analysis”, 1974). Repeated cross-sectional regression analyses are usually used to estimate the returns to the factors in such models. A time series of the vector of the regression coefficients is then used to form a covariance matrix of the factor returns.

Although we can readily observe the fundamentals of a given stock at a moment in time, and hence the stock’s exposure to the factors, the factors themselves are unobservable. For example, if we have a model in which company size is a factor, and we observe that, over a period of time, a portfolio of large-capitalization stocks outperformed a portfolio of small-capitalization stocks, we cannot tell if this outperformance was due to the company size factor alone. This is because the portfolio of large-capitalization stocks and the portfolio of small capitalization stocks may have behaved differently as a consequence of differences in factors other than capitalization. Indeed, it is likely that these two portfolios would have different values for just about any fundamental characteristic one might name. For instance, in regard to industry participation, the large-capitalization portfolio would include multinational oil companies, but the small-capitalization portfolio probably would not. Thus the difference in returns between the two portfolios might have arisen from this difference in industry exposure, not from the difference in capitalization.
The strength of “fundamental” common factor models is that they use security characteristics which are very familiar to portfolio management personnel. Such models usually also have higher in-sample explanatory power than exogenously defined models. Another advantage of such models is that, since factor exposures can be immediately observed, changes in a company’s fundamental makeup, such as a merger, will be immediately incorporated into the model. Similarly, new issues can be analyzed almost immediately. The primary criticism of endogenous common factor models is that there are often so many overlapping effects that it is nearly impossible to correctly sort them all out, making such models less effective at predicting future conditions than they are at explaining the past.

The final type of model in use today is the so-called blind factor model. In such models, the factors are not specified as being any measurable real world phenomena, but rather both the factors and the betas to those factors are inferred from the security behaviors themselves. In essence, we find those common factors that the security returns suggest must be present, even if we cannot identify the nature of the factors. Such models are estimated from the security returns using techniques such as principal components regression or maximum likelihood factor analysis. The primary benefit of such models is that, since the nature of the common factors is derived inferentially, the structure of the common factors can evolve over time to fit new conditions. Unfortunately, this is also the primary detriment, as critics argue that, without any tie to the real world, such models are unduly influenced by transitory noise in the data, resulting in unstable results. Since the common factor definitions are newly derived from each new sample of data, the user has no guarantee that the sixth factor derived from the return data set that ends today has any relation to the sixth factor derived from the return data set that ended a month ago. This may not be a problem for index funds, where we really care about “being like” our index, without regard to exactly what the criteria for similarity are. However, for active managers, the lack of definition may be problematic, as one or more of the unidentified factors may be correlated with the manager’s stock selection criteria. In such a case, efforts to control risk by managing factor exposures would limit the ability to expose the portfolio to characteristics that the manager believes to be desirable.

Irrespective of the general structure of a risk model, the key to a successful model is to understand fully the use to which a model will be put. For example, the greater the number of factors in a model, the better the ability of the model to explain past events. However, the larger number of factors means more values that must be forecast regarding the future. The more the things we must forecast, the more likely we are to err in some of those forecasts, resulting in poor forecasts of portfolio behavior. Useful discussions of these issues appear in “Some Empirical Tests in the Arbitrage Pricing Theory” by Chen and Jordan (Journal of Banking and Finance, 1993) and “A Test for the Number of Factors in an Approximate Factor Model”, by Connor and Korajczyk (Journal of Finance, 1993).
The return dispersion of individual stocks is also expressed in the implied volatility of options on equity securities. Implied volatility values typically are larger than the volatility values one would estimate for the same stock using a common factor model. One theory as to why implied volatility values are consistently higher is that stock return distributions have a slight mean reversion tendency, making the observed volatility larger as the length of the observation period becomes shorter. Since average holding periods for traded stock options are much shorter than the average holding period for an equity security, option traders would tend to use a higher volatility than investors holding the underlying stocks. The mean reversion tendency is described in Lo and McKinlay (“Stock Market Prices Do Not Follow Random Walks: Evidence from a Simple Specification Test”, Review of Financial Studies, 1988). The generally kurtotic (fat tails) nature of stock return distributions is covered in Rosenberg (“The Behavior of Random Variables with Non-Stationary Variance and the Distribution of Security Prices”, working paper University of California at Berkeley, NSF Grant 3306. 1975). Another intuitive explanation is that option traders upward bias volatility estimates to compensate for the time series variability in the instantaneous volatility.
Estimation Error in Portfolio Construction

Risk management of financial portfolios is an exercise largely based on the concepts of Modern Portfolio Theory. When we form "hedges", we are really forming portfolios whose expected variance of returns is close to zero. When we try to control the tracking error of an equity portfolio relative to some benchmark index, we are forming an “active” portfolio whose expected variance of returns is within some acceptable range of magnitude. Mean-variance optimization using the classic Markowitz method is a widely used technique. It has become the standard practice of the modern practitioner with respect to investment allocation problems, whether they are managing a traditional portfolio or hedging the risks of a trading operation.

One of the least understood aspects of the optimization process is that the inputs are interpreted as being certain. In doing an optimization using the classic Markowitz method, the required inputs to the optimizer are the means and standard deviations of the asset returns and the matrix of correlations among the asset returns. These values are the parameters of the probability distributions of the asset returns. Alternatively, one could represent the covariance matrix in the form of a factor model. Given these values any Markowitz method software can calculate an optimal portfolio.

The problem is that we do not know the required information, the parameters of the probability distributions of the asset returns. We have only our estimates of the future (and therefore not precisely known) probability distributions of asset returns. In stock portfolio allocation problems, practitioners typically create forecasts of future stock return behaviors. In general, these forecasts are derived from some analysis of the past observations, which represent a sample of the true probability distribution. We know two potential sources of forecast error. First, the observed sample of past behavior may not be a meaningful and accurate representation of the population probability distribution. Hence we have built our model on the wrong basic data. In short, the world can change over time. Second, our forecasting method may be incomplete, allowing for potential errors in the forecast, even if the observed sample parameters are a good proxy for the population parameters. It should be noted that while improved forecasting methods can somewhat mitigate the latter problem, the former always remains.

Unfortunately, the traditional Markowitz algorithm has no mechanism by which to include this second form of risk: the risk that we, the users, can simply be wrong about the input parameters. In essence, all traditional optimizers used in the traditional way get the wrong answer. The risk is consistently understated because the computer algorithm has no way to understand that we the people supplying the inputs are subject to error. The algorithm believes that if we wait long enough, our forecasts of the distributions of future stock returns will be proven exactly correct. This is a heroic and dangerous assumption that is too often overlooked.
Not only are risks persistently understated, but another result of this presumption is a severe lack of robustness in the purportedly optimal weights. This is manifested when very small changes in the input forecasts cause substantial changes in the optimal portfolio weights. This instability can also arise from the flawed presumption of certainty-equivalence of the inputs. While there was nothing wrong in Markowitz's work, the translation from theory to practice has left much to be desired.

The estimation error issue is not new. The general problem was described by Stein ("Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution", Proceedings of the 3rd Berkeley Symposium on Probability and Statistics, 1955). Later, C.B. Barry wrote "Portfolio Analysis under Uncertain Means, Variances and Covariances" (Journal of Finance, 1974). In that same year, a monograph called Estimation Risk and Optimal Portfolio Choice was published by Bawa, Brown and Klein. Although this problem was identified more than twenty years ago, practitioners have still not adopted methods to correct this flaw in their analyses.


The first way in which we may deal with estimation risk is to use a factor representation of the covariance matrix. To the extent that such a representation separates covariance into two components (a factor component believed to be persistent and a component of transitory, random effects), we may reduce the estimation errors. Users of commercially available factor models are often lulled into complacency by recognition of this point. However, no factor representation of any market is perfect. Thus, any factor model used to optimize the allocation of a portfolio or to hedge a book of trading positions will do an imperfect job. It is therefore recommended that users of commercial models do portfolio construction problems with one model, and then check the result using another independent model of the same market. Only in this way can biases in the estimation of the factor covariance matrix be discovered.

There are two well-explored methodologies for dealing with estimation risk in general. The first relies on the use of Bayesian statistical techniques. The second makes use of the methodology known as bootstrapping to consider optimal asset weights not as precise values but as probability distributions in their own right.

The more computationally efficient way to deal with estimation risk is to revise the inputs to the optimization so as to cause the Markowitz optimization algorithm to act as if the estimation risk were explicitly included in the traditional method. The most prominent work on this process of adjustment was written by Phillippe Jorion. This method is quite mathematically complex and is described in "Bayes-Stein Estimation for Portfolio
通过复杂的统计分析，历史观察到的回报率的调整因素被导出并应用于预期回报、标准差和相关系数的预测中。一般来说，预期回报会收敛到最小方差组合的预期回报。离差值会被增加，非对角相关系数估计值也会略微收敛。

一个可能的思考方式是这样的：想象一个资产分配问题，其中我们的回报预测的有效性为零。如果没有实质性差异的资产的平均预期行为，那么每个资产都相当于其他资产。我们最终会得到一个风险最小化的组合。如果我们的预测是完美的，我们就会得到传统的马科维茨过程的结果。这个"真相"在于这两个极值之间。正如我们预期的，使用贝叶斯-斯坦估计器方法创建的最优组合通常比不使用调整的比较组合更均衡。

这些差异在传统最优组合和使用贝叶斯-斯坦估配器创建的组合之间的资产权重之间有时很小，有时非常大。对主要资产类别来说，这种差异很小。对个别权益来说，特别是那些有较短或波动性历史的权益，"滑行"到最小方差组合是显而易见的。这在对冲环境中非常重要，并且合理地表明了在更复杂或异国情调证券上会有实质性的错误。

这些类型的贝叶斯调整也可以作为因素模型的构建的一部分。通常，特定资产的风险估计会被向上调整，而跨资产的风险估计的分散也会减少。调整也可以被施加到外生模型中的因素暴露或者内生模型中的因素协方差矩阵。


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potential estimation error in each of the inputs to the optimization problem. One approach to this is to use resampling methods such as "bootstrapping" on return and variance elements of historic asset (or factor) returns and "jacknifing" on the elements of the correlation matrix among asset (or factor) returns.

We begin the actual optimization process by calculating a traditional efficient frontier and selecting specific points along the frontier that correspond to specific values of the risk tolerance coefficient (slope of a tangent drawn to the frontier). We next repeat the process of calculating the efficient frontier 1000 times, each time randomly perturbing the input information by magnitudes derived from the resampling or substituted processes. By observing the same points along each frontier, we have 1001 points for each value of the risk tolerance coefficient that we had selected along the original frontier. In essence, each known point along the efficient frontier is now represented by a region consisting of 1001 related points (related by having the same risk tolerance coefficient). We next calculate the Euclidean distance between each of the additional 1000 points and the original point. To establish a confidence interval, we then eliminate the 10% of the 1000 points that are most distant from the original optimal portfolio. The remaining 900 points represent a region in mean-variance space within which we are indifferent. Given the admitted uncertainty of the inputs, we are indifferent from a risk/return standpoint to holding any portfolio that falls within the region. This indifference concept can be used to eliminate unproductive transaction costs, as we can now determine when a rebalancing is really needed. We can also calculate the composition of a portfolio that falls at the density center of the region; this portfolio composition being least likely to be driven outside the region by estimation error in the inputs.

The empirical result of such work has been very intuitively appealing. In short, the "density center" portfolios tend to be close to the original optimal portfolios but slightly biased toward equal weighting of the assets. Consider a spectrum with traditional optimization at one extreme, where all input information is considered certain. At the other extreme, all input information is devoid of any meaningful predictive content. In the latter case, where no information is meaningful, all assets are fungible and an equal-weighted portfolio will result (or equal-active-weighted relative to a benchmark). By formally introducing estimation error into the process, we change the portfolio composition toward equal weighting. This density-center portfolio will be very similar to the portfolio produced by the Bayes-Stein estimator methods described earlier. The advantage of the fuzzy frontier approach is we can now look at the weighting of each asset as a probability distribution in its own right. A similar, but more analytic, approach to this problem is presented in “Computing Efficient Frontiers with Estimated Parameters” by Mark Broadie (Annals of Operations Research, 1993).

Estimation risk is a very real problem for those having the responsibility for risk management of financial portfolios. For certain assets, the estimation risks are actually larger than the intrinsic risks of the assets. One such example, real estate properties, is
pointed out in "Why the Efficient Frontier for Real Estate is Fuzzy" by Richard Gold (Journal of Real Estate Portfolio Management, 1995).

Conventional implementations of MPT consistently underestimate the risks and result in portfolio combinations that are not robust and that are typically too heavily concentrated in too few assets. For cross-hedging trading risks, the problem is further magnified, as the typical number of assets is small, limiting the extent to which the Central Limit Theorem can be relied upon to save us from ourselves.

**Conclusion**

Understanding and controlling risks of equity investing is among the most challenging aspects of the portfolio management process. We must first come to a definition of risk which suits our needs. Assuming we choose the conventional definition of risk, sophisticated models are available to help in the forecasting of risk, both at the portfolio level and for individual securities.

Perhaps the aspect of the problem where current practices require the greatest improvement is in recognizing that even the most sophisticated approaches to risk management rely on assumptions that may not prove to be reliable. We therefore must apply another level of effort to compensate for inadequacies in the risk estimation process itself.