Diversification Return and Leveraged Portfolios

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Introduction

» Portfolio rebalancing is a common practice: simple yet beneficial

» Leveraged portfolios are now common

» Previous literatures \((\text{Booth and Fama 1992})\) and \((\text{Willenbrock 2011})\) on long-only unlevered portfolio

» Personal interests
  - Simple but illustrative model
  - Risk Parity portfolios
  - Mathematical certainty in finance?
It generates diversification return

A simple example – maybe the best example

<table>
<thead>
<tr>
<th></th>
<th>Year 1</th>
<th>Year 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment 1</td>
<td>100%</td>
<td>-50%</td>
</tr>
<tr>
<td>Investment 2</td>
<td>-50%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Both investments have 25% arithmetic average return

Both investments have 0% geometric return after two years but they have perfect negative correlation
It generates diversification return

» Consider a 50/50 portfolio

<table>
<thead>
<tr>
<th>Portfolio Rebalance</th>
<th>Year 1</th>
<th>Rebalance</th>
<th>Year 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$50</td>
<td>$100</td>
<td>$62.5 (50%)</td>
<td>$31.25</td>
</tr>
<tr>
<td>$50</td>
<td>$25</td>
<td>$62.5 (50%)</td>
<td>$125</td>
</tr>
<tr>
<td><strong>Total $100</strong></td>
<td><strong>$125 (25%)</strong></td>
<td><strong>$125</strong></td>
<td><strong>$156.25 (25%)</strong></td>
</tr>
</tbody>
</table>

» It generates 25% each year

» But one must **rebalance to the original weights!**
Diversification return of long-only portfolios

» Sell winners and buy losers – **mean-reverting strategy**

» The diversification return is always positive
**Long-short portfolios**

» Consider a 200/100 portfolio

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 1</th>
<th>Rebalance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$200</td>
<td>$400</td>
<td>114%</td>
</tr>
<tr>
<td>-$100</td>
<td>-$50</td>
<td>-14%</td>
</tr>
<tr>
<td>Total $100</td>
<td>$350 (250%)</td>
<td>100%</td>
</tr>
</tbody>
</table>

» To rebalance, one buys winners and sell losers – trend-following or momentum strategy
Diversification return of long-short portfolios

» Buy winners and sell losers

» Is the diversification return negative then?

» Is momentum a losing proposition?

<table>
<thead>
<tr>
<th></th>
<th>Year 1</th>
<th>Year 1</th>
<th>Rebalance</th>
<th>Year 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$200</td>
<td>$400</td>
<td>114%</td>
<td>$700 (200%)</td>
<td>$350</td>
</tr>
<tr>
<td>-$100</td>
<td>-$50</td>
<td>-14%</td>
<td>-$350 (-100%)</td>
<td>-$700</td>
</tr>
<tr>
<td>Total $100</td>
<td>$350(250%)</td>
<td>100%</td>
<td>$350</td>
<td>-$300</td>
</tr>
</tbody>
</table>
Portfolio Rebalance

Leveraged portfolios

» Hedge fund/Private equity/Real estate/Risk Parity

» Leveraged ETFs: 2X, 3X

» Reverse ETF: -X, -2X

» What happens to diversification returns of these portfolios?

» Does rebalancing sometimes hurt performance?
Diversification Return

Definition

» Arithmetic average return

\[
\mu = \frac{1}{N} \sum_{i=1}^{N} r_i
\]

» Geometric average return

\[
1 + g = \prod_{i=1}^{N} \left(1 + r_i\right)^{1/N}
\]

» Geometric average is less than arithmetic average

\[
g \approx \mu - \frac{1}{2} \sigma^2
\]
Diversification Return

Definition

» Diversification return definition

\[ r_d \equiv g_p - \sum_{i=1}^{N} w_i g_i \]

» Portfolio geometric return in excess of average geometric return

» The whole is bigger than sum of its parts if diversification return is positive
The key formula

Diversification return

\[ r_d \equiv g_p - \sum_{i=1}^{N} w_i g_i \]

Diversification return in terms of volatilities (Willenbrock 2011)

\[ r_d = \mu_p - \frac{1}{2} \sigma_p^2 - \sum_{i=1}^{N} w_i \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) \]

\[ r_d = \frac{1}{2} \left( \sum_{i=1}^{N} w_i \sigma_i^2 - \sigma_p^2 \right) \]

Diversification return is one half of the weighted average of individual variances minus the portfolio variance
Applications

Portfolio examples

» Two-asset portfolio with one risky asset
» Leveraged, inverse ETFs
» Portfolio with two risky assets
» Long-only unlevered portfolios
» Leveraged Risk Parity portfolios
Applications

Portfolio with one risky asset

- One risky asset \((w)\) and risk-free asset

\[
    r_d = 0.5(w - w^2)\sigma^2
\]

\[
    r_d = \frac{1}{2} \left( \sum_{i=1}^{N} w_i \sigma_i^2 - \sigma_p^2 \right)
\]
Applications

Ultra long and inverse ETFs

» Match index return multiples daily, i.e. 2X, 3X, -X, -2X
  » A single risky asset – cap-weighted indices don’t rebalance

» Rebalance on a daily basis to the fixed weight
  » Trending-following at the close/Liquidity demanding

» They don’t match index return multiple over longer period

» Significant negative diversification returns
Ultra long and inverse ETFs

» Annual volatility of 23%, daily volatility of 1.45% \( \frac{23}{\sqrt{250}} \)

\[
r_d = 0.5(w - w^2) \cdot (1.45\%)^2
\]

» Return slippage

<table>
<thead>
<tr>
<th></th>
<th>-3X</th>
<th>-2X</th>
<th>-1X</th>
<th>2X</th>
<th>3X</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily</td>
<td>-0.1%</td>
<td>-0.1%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>-0.1%</td>
</tr>
<tr>
<td>Annual</td>
<td>-27.6%</td>
<td>-14.9%</td>
<td>-5.2%</td>
<td>-5.2%</td>
<td>-14.9%</td>
</tr>
</tbody>
</table>
Ultra long and inverse ETFs

- SSO – ultra long, SDS – ultra short
- Flat year in 2011 for the index

<table>
<thead>
<tr>
<th></th>
<th>SP 500</th>
<th>SSO (2X)</th>
<th>SDS (-2X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiples</td>
<td>2.11%</td>
<td>4.22%</td>
<td>-4.22%</td>
</tr>
<tr>
<td>Slippage</td>
<td>-5.2%</td>
<td>-14.9%</td>
<td></td>
</tr>
<tr>
<td>Prediction</td>
<td>-1.24%</td>
<td>-18.51%</td>
<td></td>
</tr>
<tr>
<td>Actual</td>
<td>-3.45%</td>
<td>-18.94%</td>
<td></td>
</tr>
</tbody>
</table>
Applications

Portfolio with two risky assets

» Correlation is now important

\[ r_d = \frac{1}{2} \left( w_1 \sigma_1^2 + w_2 \sigma_2^2 - w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 - 2 \rho w_1 w_2 \sigma_1 \sigma_2 \right) \]

» For long-only portfolios, low correlations leads to high diversification returns

\[ r_{d,\text{max}} = \frac{1}{2} \left[ w_1 \sigma_1^2 + w_2 \sigma_2^2 - (w_1 \sigma_1 - w_2 \sigma_2)^2 \right] \]

\[ r_{d,\text{min}} = \frac{1}{2} \left[ w_1 \sigma_1^2 + w_2 \sigma_2^2 - (w_1 \sigma_1 + w_2 \sigma_2)^2 \right] \]
Applications

Portfolio with two risky assets

- Asset volatility 5%, and 20%, different correlations
Applications

Long-only unlevered portfolio

» Diversification return is always non-negative

\[ r_d = \frac{1}{2} \left( \sum_{i=1}^{N} w_i \sigma_i^2 - \sigma_p^2 \right) \geq \frac{1}{2} \left[ \sum_{i=1}^{N} w_i \sigma_i^2 - \left( \sum_{i=1}^{N} w_i \sigma_i \right)^2 \right] \]

» It can be proven mathematically that the right-hand side is always non-negative
Leveraged long-only portfolios

- What happens when one rebalance a leveraged long-only portfolio?
  - Mean-reverting across different assets (sell winners and buy losers)
  - Trend-following at the portfolio level (de-lever with loss and re-lever with gain) – similar to portfolio insurance

- Its diversification return is a tug of war between the two
  - Leverage increases positive diversification return across assets
  - Leverage produces negative diversification return on top
Leveraged long-only portfolios

» Rescaled the portfolio to an unlevered portfolio

\[ \sum_{i=1}^{N} w_i = L, \]

\[ w_i^s = \frac{w_i}{L}, \sum_{i=1}^{N} w_i^s = 1. \]

» The rescaled portfolio has diversification return \( r_d^s \) and volatility \( \sigma_s \)

» The diversification return of the levered portfolio is

» Mutual fund separation theorem

\[ r_d = L \cdot r_d^s + \frac{1}{2} \left( L - L^2 \right) \sigma_s^2 \]
## Applications

### Leveraged long-only portfolios

» From 60/40 to Risk Parity portfolios

<table>
<thead>
<tr>
<th></th>
<th>( \rho = -0.5 )</th>
<th>( \rho = 0 )</th>
<th>( \rho = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>60/40</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diversification return</td>
<td>0.63%</td>
<td>0.51%</td>
<td>0.39%</td>
</tr>
<tr>
<td>Portfolio volatility</td>
<td>11.1%</td>
<td>12.2%</td>
<td>13.1%</td>
</tr>
<tr>
<td><strong>20/80</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diversification return</td>
<td>0.42%</td>
<td>0.34%</td>
<td>0.26%</td>
</tr>
<tr>
<td>Portfolio volatility</td>
<td>4.00%</td>
<td>5.66%</td>
<td>6.93%</td>
</tr>
<tr>
<td><strong>Risk Parity</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diversification return</td>
<td>0.77%</td>
<td>0.34%</td>
<td>0.08%</td>
</tr>
<tr>
<td>Portfolio volatility</td>
<td>11.1%</td>
<td>12.2%</td>
<td>13.1%</td>
</tr>
<tr>
<td>Portfolio leverage</td>
<td>278%</td>
<td>215%</td>
<td>190%</td>
</tr>
</tbody>
</table>
Leveraged long-only portfolios

» Risk Parity portfolios with different correlations
Summary

» Simple portfolio rebalancing generates diversification returns

» Across assets, rebalancing means mean-reverting

» For leveraged portfolios, top level rebalancing means trend-following
  » Stop-loss is just mechanical

» Long-only unlevered portfolios always have positive diversification returns
Summary

» Levered and/or inverse ETFs have significant negative diversification returns

» Leveraged long-only portfolios’ diversification return consists of two parts
  » Across asset diversification return magnified by leverage - positive
  » Top-down leverage - negative

» The benefit of low correlations: better diversifications, higher diversification returns