No-arbitrage condition and expected returns when assets have different betas in up and down markets

Peter Xu
Quantitative Management Associates, LLC

Northfield Annual Conference

October 7, 2013

Recent research on returns and downside risk

Post, Van Vliet and Lansdorp 2012)
Ang, Chen and Xing (2006)
Estrada (2007)
Pedersen and Hwang (2007)
Kaplanski (2004)

$$E(R_i) = \underset{(3.39)}{0.044} + \underset{(7.17)}{0.069^* \beta^-} - \underset{(-4.85)}{0.029^* \beta^+}$$

Source: Ang, Chen and Xing (***Review of Financial Studies*** 2006).

Expected Return on Equity Index Options

Coval and Shumway (2001)

X-S	-5 to 0	0 to 5
Weekly SPX Call Option Returns		
mean	1.85	2.00
t-stat	(0.66)	(0.55)
median	-4.46	-9.55
mean BS β	31.20	40.02
Weekly SPX Put Option Returns		
mean	-9.50	-7.71
t-stat	(-3.05)	(-2.81)
median	-21.57	-15.06
mean BS β	-35.23	-31.11

Source: Coval and Shumway (***Journal of Finance*** 2001)

The Model

Return-generating process:

$$R_i = \alpha_i + \beta_{i+} R_{m+} + \beta_{i-} R_{m-} + e_i$$

No-arbitrage condition

For any w_i that satisfy:

$$\sum w_i \leq 0$$
$$\sum w_i \beta_{i+} \geq 0$$
$$\sum w_i \beta_{i-} \leq 0$$
$$\sum w_i e_i \approx 0$$

we must have

$$\sum w_i \alpha_i \leq 0$$

It can be proved that there exist $\phi_0>0$, $\phi_+>0$, and $\phi_->0$ such that for any stock i,

$$\alpha_i = \phi_0 - \phi_+ \cdot \beta_{i+} + \phi_- \cdot \beta_{i-}$$

What are the values of ϕ_0, ϕ_+ and ϕ_-?

Riskless asset:

$$R_i = R_f + 0 \cdot R_{m+} + 0 \cdot R_{m-} \quad \rightarrow \quad \phi_0 = R_f$$

At-the-money call option on market index:

$$R_i = -1 + \frac{S}{C} \cdot R_{m+} + 0 \cdot R_{m-} \quad \rightarrow \quad -1 = R_f - \phi_+ \cdot \frac{S}{C}$$

At-the-money call option on market index:

$$R_i = -1 + 0 \cdot R_{m+} - \frac{S}{P} \cdot R_{m-} \quad \rightarrow \quad -1 = R_f - \phi_- \cdot \frac{S}{P}$$

Equilibrium trade-off

$$\alpha_i = R_f - \frac{C(1+R_f)}{S}\cdot\beta_{i+} + \frac{P(1+R_f)}{S}\cdot\beta_{i-}$$

Equilibrium expected returns

$$E(R_i) = R_f - \frac{C(1+R_f) - S \cdot E(R_{m+})}{S} \cdot \beta_{i+} + \frac{P(1+R_f) + S \cdot E(R_{m-})}{S} \cdot \beta_{i-}$$

Expected returns using risk-neutral probabilities

$$E(R_i)= R_f + (E(R_{m+}) – E'(R_{m+}))\cdot\beta_{i+} + (E(R_{m-}) - E'(R_{m-}))\cdot\beta_{i-}$$

Interpretation of risk premiums

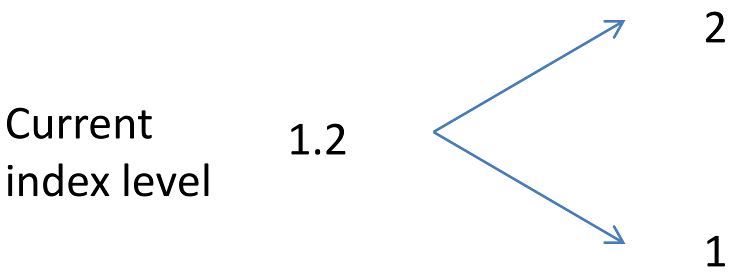
$$E(R_i) = R_f - \frac{C(1+R_f) - S \cdot E(R_{m+})}{S} \cdot \beta_{i+} + \frac{P(1+R_f) + S \cdot E(R_{m-})}{S} \cdot \beta_{i-}$$

Investor preferences and risk premiums on upside and downside betas

$$E(R_i) = R_f + RP_+ \cdot \beta_{i+} + RP_- \cdot \beta_{i-}$$

RP_+	< 0	if investors as a whole are lottery buyers;
	> 0	if investors as a whole are hedgers.
RP_-	> 0	

Option prices in a two-state world



A portfolio of 1 share of index and - 1.25 at-the-money calls has a fixed future payoff of 1

The price of the call option is given by:

1.2-1.25*C=1
C=0.16

Returns on index options and risk premiums on upside and downside betas

Index Level in Good State	2	2	2	2
Index Level in Bad State	1	1	1	1
Probabilities of Good/Bad states	0.5/0.5	0.5/0.5	0.9/0.1	0.9/0.1
Index Level	1.20	1.36	1.73	1.86
Exp. Return on Index	25%	10%	10%	5%
Price of Call	0.16	0.23	0.20	0.15
Exp. Return on Call	150%	37.5%	23.8%	11.2%
Price of Put	0.16	0.23	0.20	0.15
Exp. Return on Put	-37.5%	-21.4%	-63.3%	-47.5%
Predicted Risk Premium on β_+	**20%**	**6%**	**3%**	**1%**
Predicted Risk Premium on β_-	**5%**	**4%**	**7%**	**4%**

Data and empirical results

Time Interval: 1985 – 2012

Universe: Russell 3000 at beginning of each year

Methodology: β_+ and β_- are estimated from daily returns in a year against an EW average of all stocks.

Two measures of contemporaneous returns are used: total return and arithmetic mean of daily returns in a year.

Sample statistics on upside and downside beta groups

Rank for upside beta		Rank for downside beta: 1(low)	2	3	4(high)	All
1(low)	β_+	0.3	0.3	0.2	-0.0	0.2
	β_-	0.4	0.7	1.1	2.0	0.8
	#Obs.	8,918	5,048	3,070	2,387	19,423
2		0.7	0.7	0.7	0.7	0.7
		0.4	0.8	1.1	1.8	0.9
		5,415	6,701	4,773	2,548	19,437
3		1.1	1.1	1.1	1.2	1.1
		0.3	0.8	1.1	1.8	1.0
		3,113	5,223	6,461	4,646	19,443
4(high)		2.1	1.8	1.8	2.0	2.0
		-0.0	0.8	1.1	1.9	1.4
		1,977	2,465	5,139	9,852	19,433
All		0.7	0.9	1.1	1.4	1.0
		0.3	0.8	1.1	1.9	1.0
		19,423	19,437	19,443	19,433	77,736

Average contemporaneous returns and upside and downside betas

Rank for upside beta	Rank for downside beta 1(low)	2	3	4(high)	All
1(low)	R_{yr}(%) 9.0 R_{day}(%) 0.039	10.4 0.045	9.8 0.043	6.6 0.043	9.2 0.042
2	11.5 0.048	13.5 0.056	14.4 0.059	16.7 0.065	13.6 0.065
3	8.7 0.042	11.9 0.054	14.5 0.064	18.5 0.077	13.8 0.061
4(high)	-4.5 0.019	9.9 0.045	8.4 0.045	17.2 0.078	11.7 0.059
All	8.3 0.040	11.8 0.051	12.1 0.055	16.2 0.072	12.1 0.054

Regression results

	Intercept	β_+	β_-	R^2
• annual returns	6.1 (13.6)	-0.7 (-2.1)	6.6 (18.8)	0.0047
• average daily returns	0.032 (22.1)	0.002 (1.9)	0.020 (17.4)	0.0046

Conclusions

- Upside and downside betas are priced separately, and their risk premiums are related to the difference between price and expected payoff of options on the market;

- Stock returns are strongly and positively correlated with downside betas but weakly correlated with upside betas;

- The risk premiums on upside and downside betas are consistent with observed returns on stock index options;

- These results imply that investors are concerned about low-probability but significant downside events.