
A Case for Arithmetic Attribution

A Good-Humored Ramble through the Forests of Two Continents

Mark R. David, CFA



Why bother? Everyone already knows that...

- Returns compound
- Time-period linking is a dodgy kludge that arbitrarily distorts analytic meaning
- Americans prefer simple, tidy sums to precise analysis
- Ease-of-use hardly constitutes a cogent argument

Definition: Management Effect

- The geometric formulation:

$$\frac{1 + R_{P,t}}{1 + R_{B,t}} - 1$$

- The arithmetic formulation:

$$R_{P,t} - R_{B,t}$$

Attendez! Management Effect is not Attribution.

Bacon's Three Advantages of the Geometric Management Effect (*Excess Return*)

- Proportionality
- Convertibility
- Compoundability

Proportionality:

“...client is most concerned about the value of the portfolio at the end of period (not the start), and quite rightly should ask how much larger is the portfolio now, than it would have been, had the portfolio been invested in the benchmark.”

- I quite agree, with the possible exception of the word “most”.
- There are other valid points of view, e.g.
 - *“How much larger is the portfolio now, compared to the benchmark, as a percent of my initial commitment?”*

Both are valid; why discard one in favor of the other?

Convertibility:

“By far the most convincing argument for using Geometric Excess Returns is their convertibility across different currencies.”

- Demonstrated using an arithmetic example that naively combines local currency and FX returns.

Just don't do that.

Compoundability:

“Geometric Excess Returns are also compoundable over time.”

- True, insofar as the total Management Effect takes us...

But what about Geometric Attribution?

Interaction: *a brief digression...*

- Supposedly a “feature” of Brinson methodologies...
- Actually, it occurs in any management process with parallel steps
- Several geometric methodologies claim to eliminate interaction
- Using algebra to combine interaction into another attribution effect does not eliminate it

Interaction Happens

Tying Out: A Criterion for Attribution

- What we “want” to see in an attribution report are meaningful numbers that **explain** what happened: namely, the fact that portfolio and benchmark returns are different
- ***Tying out*** is ***explaining*** – 2 or more explanatory numbers that relate algebraically to an explained number
- Tying out can be of either flavor:
 - Arithmetic: $a + b = c$
 - Geometric $(1 + a) * (1 + b) = 1 + c$

When attribution effects don't tie out, they don't explain

Dimensionality of Tie-Outs

	Portfolio Contribution	Benchmark Contribution	Management Effect	Attribution Effect 1	Attribution Effect n	Attribution Effect N
Segment 1	$C_{1,p,t}$	$C_{1,b,t}$	$M_{1,t}$	$A_{1,1,t}$	$A_{n,1,t}$	$A_{N,1,t}$
Segment s	$C_{s,p,t}$	$C_{s,b,t}$	$M_{s,t}$	$A_{1,s,t}$	$A_{n,s,t}$	$A_{N,s,t}$
Segment S	$C_{S,p,t}$	$C_{S,b,t}$	$M_{S,t}$	$A_{1,S,t}$	$A_{n,S,t}$	$A_{N,S,t}$
Total	$R_{p,t}$	$R_{b,t}$	M_t	$A_{1,t}$	$A_{n,t}$	$A_{N,t}$

And that's just for single-period attribution! We also want each of the above numbers to tie-out to its multi-period result.

Tying Out: the Scorecard

	Tie-out Relationship	MCNP	BKT (top down) / Bacon	Mencherero	Shestopaloff	Reztsov Geometric pG ⁵
Single-period Tie-outs	Segment → Portfolio: Contribution	Σ	Σ	Σ	Σ	Σ
	Segment: Contribution → Management Effect	Σ	Π ¹	Π ¹	Π ¹	Σ
	Portfolio: Contribution → Management Effect	Σ	Π	Π	Π	Σ
	Segment → Portfolio: Management Effect	Σ	None ¹	None ¹	None ¹	Σ
	Segment → Portfolio: Attribution Effects	Σ	Σ	Π ²	Σ	Σ
	Segment: Attribution Effects → Management Effect	Σ	None ¹	None ^{1,3}	None ¹	Σ
	Portfolio: Attribution Effects → Management Effect	Σ	Π	Π ²	Π	Σ
Single → Multi-period Tie-outs	Segment: Management Effect	Σ	Π	Π	None ⁴	Σ
	Segment: Attribution Effects	Σ	Π	Π	None ⁴	Σ
	Portfolio: Management Effect	Σ	Π	Π	None ⁴	Σ
	Portfolio: Attribution Effects	Σ	Π	Π	None ⁴	Σ
Multi-period Tie-outs	Segment → Portfolio: Management Effect	Σ	None ¹	None ¹	None ⁴	Σ
	Segment → Portfolio: Attribution Effects	Σ	None	Π ²	None ⁴	Σ
	Segment: Attribution Effects → Management Effect	Σ	None ¹	None ^{1,3}	None ⁴	Σ
	Portfolio: Attribution Effects → Management Effect	Σ	Π	Π ²	None ⁴	Σ

Observations and Commentary

- Observation: No geometric method* ties out consistently and completely
- Commentary:
 - α : A mixture of arithmetic, geometric and non- tie-outs on the same page guarantees confusion
 - β : Everyone agrees that segment contributions sum to return
 - $\alpha + \beta \rightarrow$ Geometric attribution is guaranteed to confuse
 - “None” is a failure to explain results
 - Sector attribution effects don’t explain management effect
 - Sector management effects don’t explain total

**Reztsov ties out arithmetically, we’ll touch on this in a bit.*

Kludges in Search of Tie-outs

- Various: “correction factor”, “adjustment factor”, “symmetry divisor”, “period coefficient”
- Achieves desired algebraic relationship (e.g., “compoundability”) at the expense of accurate, meaningful explanation of management process
- Most arithmetic method methods equally guilty
 - Arithmetic adjusts single- to multi-period
 - Geometric adjusts sector- to total- level

Where did we go wrong? How can we meaningfully measure and explain the management process?

Reverting to a More Basic Measure

- Much of the struggle and complexity introduced by attribution methods arises from the attempt to combine and tie out incompatible input quantities into a consistent analysis.
 - Returns vs. contributions vs. effects
 - Snapshot quantities (weight) vs. periodic quantities (return)
- Segment-level returns and weights themselves are derived from precursor formulas, of which many versions exist.

Let's begin with a more basic, comprehensible and unambiguous quantity: Amounts of money.

Money Compounded Notional Portfolios (MCNP)

	Portfolio Profit	Notional Benchmark Profit	Management Effect	Notional Allocation Effect	Notional Selection + Interaction Effects
Segment 1	$P_{1,t}$	$B_{1,t}$	$P_{1,t} - B_{1,t}$	$A_{1,t} - B_{1,t}$	$P_{1,t} - A_{1,t}$
Segment n	$P_{n,t}$	$B_{n,t}$	$P_{n,t} - B_{n,t}$	$A_{n,t} - B_{n,t}$	$P_{n,t} - A_{n,t}$
Segment N	$P_{N,t}$	$B_{N,t}$	$P_{N,t} - B_{N,t}$	$A_{N,t} - B_{N,t}$	$P_{N,t} - A_{N,t}$
Total	P_t	B_t	$P_t - B_t$	$A_t - B_t$	$P_t - A_t$

- All quantities are amounts of money profited
- Notional Benchmark, Allocation and Selection portfolios are all constructed to be the same size as beginning-of-period portfolio
- Sum single period to multi-period results
- Not specific to Brinson methodologies, can be extended to any notional portfolio method
- Equivalent to Davies, Owen, Laker (2001) as augmented by David (2006)

Advantages of MCNP

- To explain arithmetic management effect, simply divide all those amounts by the size of the portfolio at beginning-of-period
 - They become – consistently – contributions to the arithmetic Management Effect
- What could be easier to understand and less ambiguous than contributions?
 - To the total Management Effect
 - From notional benchmark and attribution portfolios which are easily and unambiguously constructed at start of period
- Ties out arithmetically in all dimensions

Critique: Whatever Happened to Proportionality?

- Geometric Management Effect: “... *how much larger is the portfolio now, than it would have been, had the portfolio been invested in the benchmark?*”
- If (as specified) notional Benchmark is same size as portfolio at beginning of period ($V_{P,0} = V_{B,0}$), then Geometric Management Effect is:

$$M_{0:t} = \left(\frac{1 + R_{P,0:t}}{1 + R_{B,0:t}} \right) - 1 = \frac{R_{P,0:t} - R_{B,0:t}}{1 + R_{B,0:t}} = \frac{P_{0:t} - B_{0:t}}{V_{B,0}(1 + R_{B,0:t})} = \frac{P_{0:t} - B_{0:t}}{V_{B,t}}$$

- Therefore, simply divide all money amounts by the end size of the notional benchmark.
 - They become – consistently – contributions to the geometric Management Effect

Critique: What About External Flows?

- Explicitly not addressed in article, for length
- Three simple ways of handling:
 - Scale profit amounts in periods $t > 1$ by effect of net contributions
 - Scale portfolio and notional portfolios to common, arbitrary size – e.g., \$1 – at beginning of each period
 - Most practical: use weights and returns, but scale $t > 1$ returns by cumulative indexed portfolio (notional) returns from 1 to $t-1$.

Much Worse Critique

INPUTS		Portfolio		Benchmark	
		Weight	Return	Weight	Return
Period 1	Sector A	33.33%	60%	33.33%	10%
	Sector B	33.33%	0%	33.33%	10%
	Sector C	33.33%	0%	33.33%	10%
Period 2	Sector A	33.33%	0%	33.33%	10%
	Sector B	33.33%	60%	33.33%	10%
	Sector C	33.33%	0%	33.33%	10%
Period 3	Sector A	33.33%	0%	33.33%	10%
	Sector B	33.33%	0%	33.33%	10%
	Sector C	33.33%	60%	33.33%	10%

OUTPUT	3-period Geometric Management Effect – MCNP	3-period Geometric Management Effect – All Others
Sector A	6.74%	8.76%
Sector B	9.74%	8.76%
Sector C	13.35%	8.76%

The Stages of Grief

- Denial, Bargaining & Depression
- Conditioned by years of literature...
- Acceptance (perhaps Rationalization?)
 - These are contributions (not returns)
 - Periodicity as an artifact of performance systems
 - Passive investment and periodicity

These Are Contributions...

- ...to the 3-period, total geometric management effect
- Of course – trading & changes in total and relative sector sizes occurred during the 3-period analysis
- Contributions tie-out; they explain the total, 3-period management effect

OUTPUT	3-period Geometric Management Effect – MCNP	3-period Geometric Management Effect – All Others
Sector A	6.74%	8.76%
Sector B	9.74%	8.76%
Sector C	13.35%	8.76%
Total	29.83%	29.83%

Neither sums nor compounds

Who Said the Investment Process Is Periodic?

- Systems designers, that's who.
- Investment managers don't rebalance instantaneously, once at beginning of period.
- If you treat the 3-period example as a single period, with transaction-based performance:

OUTPUT	3-period Geometric Management Effect – MCNP	3-period Geometric Management Effect – All Others
Sector A	6.74%	8.08%
Sector B	9.74%	11.68%
Sector C	13.35%	16.00%

“Passivity” - Relative to the Analysis Period?

- Example is a classic “passive allocation” extreme.
- Nonetheless, a lot of trading happens during that analysis period.
- What if the manager chose the same starting point, then went to the Caribbean for 3 periods?

OUTPUT	3-period Geometric Management Effect – MCNP	3-period Geometric Management Effect – All Others
Sector A	6.74%	8.76%
Sector B	6.74%	5.74%
Sector C	6.74%	3.58%

Final Scorecard

- Management Effect
 - Geometric proportionality: arithmetic can do that
 - Geometric convertibility: red herring
 - Compoundability: works at the total level, but...
 - Attribution
 - Results should meaningfully and accurately measure the investment process; arbitrary adjustment to suit other goals subverts this goal.
 - Tying out: failure to consistently do so is not only confusing, but a failure to explain what happened.
 - Money (contribution) attribution: ???
-

References

- Bacon, Carl, “Excess Returns – Arithmetic or Geometric?”, *The Journal of Performance Measurement*, Spring 2002, Vol. 6 No. 3, pp. 23-31
- Burnie, Stephen J., Knowles, James A., and Teder, Toomas J., “Arithmetic and Geometric Attribution”, *The Journal of Performance Measurement*, Fall 1998, Vol. 3 No. 1, pp. 59-68
- Brinson, Gary P., Hood, L. Randolph, and Beebower, Gilbert L., “Determinants of Portfolio Performance”, *Financial Analysts Journal*, July-August 1986, pp. 39-44
- David, Mark R., “Sector-level Attribution Effects with Compounded Notional Portfolios”, *The Journal of Performance Measurement*, Spring 2006, Vol. 10 No. 3, pp. 40-54
- David, Mark R., “Bespoke Attribution: Illustrating the Manager’s Process”, *The Journal of Performance Measurement*, Winter 2009/2010, Vol. 14 No. 2, pp. 12-26
- Davies, Owen and Laker, Damien, “Multiple-period Performance Attribution Using the Brinson Model,” *The Journal of Performance Measurement*, Fall 2001, Vol. 6 No. 1, pp. 12-22
- Menchero, Jose G., “A Fully Geometric Approach to Performance Attribution”, *The Journal of Performance Measurement*, Winter 2000/2001, Vol. 5 No. 2, pp. 22-30
- Reztsov, Andrei, “Geometric and Arithmetic Approaches to Attribution Linking are Equivalent”, *The Journal of Performance Measurement*, Winter 2011/2012, Vol. 16 No. 2, pp. 46-56
- Shestopaloff, Yuri, “A Geometric Attribution Model and a Symmetry Principle”, *The Journal of Performance Measurement*, Summer 2008, Vol. 12 No. 4, pp. 29-39

Mark R. David, CFA
Essex River Analytics
mark.david@essexriver.com
(01) 978-815-5443
www.essexriver.com

