Optimization with Composite Assets Using Implied Covariance Matrices

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27th December 1998

Introduction

A very common but difficult problem in quantitative portfolio management is the inclusion of composite assets (e.g., a futures contract on a stock index) into a portfolio optimization framework that relies on a linear factor risk model. This paper proposes to overcome the difficulties by transforming the problem to the equivalent full-covariance matrix problem (see Markowitz). The elements of the covariance matrix used are implied from the linear factor model.

Linear factor models of security returns and risk are widely used in current portfolio management. The principal advantage of linear factor models over the use of full covariance matrices is the ability to separate historically observed security variances and covariances into two portions. The first portion is presumed to arise from structural aspects of the security markets and are referred to as common factor risks. We presume that security risks arising from common factors will persist into the future. Linear factor models also assume that security-specific risk also exists. By construction, security-specific risks are presumed to be uncorrelated across any two distinct securities. A review of the literature with respect to modeling the risk of equity security portfolios is provided in diBartolomeo (1997).

Linear factor risk models express the expected covariance matrix of security returns in the form of a factor covariance matrix to which each security is exposed and a security-specific portion. The usual mathematical formulation is:

$$V_p = \text{SUM}\ [_{i=1\ \text{to}\ n}\ \text{SUM}\ [_{j=1\ \text{to}\ n}\ E_{pi}\ E_{pj}\ S_i\ S_j\ R_{ij}\ ]] + \text{SUM}\ [_{k=1\ \text{to}\ m}\ W_k^2\ S_k^2] \qquad (1)$$

$$E_{pi} = \text{SUM}\ [_{k=1\ \text{to}\ m}\ W_k * B_{ki}] \qquad (2)$$

Where $V_{p\,=}$ variance of portfolio return
- n = number of factors in the risk model
- m = number of securities in the portfolio
- E_{pi} = exposure of the portfolio to factor i
- S_i = standard deviation returns attributed to factor i
- R_{ij} = correlation between returns to factor i and factor j
- W_k = weight of security k in the portfolio
- S_k = standard deviation of security specific returns for security k
- B_{ki} = beta of security k to factor I

One of the key assumptions of this formulation regards the item called security-specific returns. As is implied by the terminology, it is assumed that such returns are indeed specific to a given security. Hence they are therefore uncorrelated between each possible pairing of security specific returns within the portfolio. We are assuming that the factors specified in the model are sufficient to explain all covariance among the securities. In this sense, we presume that the factor specification is *complete*. We should note that completeness is used in a slightly different context than in most academic literature, that uses completeness to refer to the number of factors necessary to describe the equilibrium relationship between risk and return. In our case, we make no assumptions about equilibrium but rather about the exhibition of no pairwise correlation in residual returns.

The Problem of Composite Assets

The presumption that security-specific risks are uncorrelated fails, once we introduce a composite asset that consists of a set of underlying securities in some proportions. Consider the situation where we are trying to measure the risk of holding a composite asset consisting of the members of the Standard & Poors 500 stock index. At the same time, we short-sell the 500 actual stocks in the index in the appropriate proportions. In actuality, the risk of such a position is zero. A linear factor model would correctly measure that the common factor risks were zero.

However, the computed security-specific risks would not be zero. The model would perceive that this portfolio had a net long position in one asset (the composite), while having a net-short position in the 500 stocks, rather than a net zero position in all assets. The security-specific risks of these five hundred and one active positions would be presumed uncorrelated and not sum to zero. The magnitude of the computed security-specific risk of this portfolio might be large or small, but it remains obviously wrong. It should be noted that if the composite asset were an actual future or option contract on the index, there would also be basis risk in the position that would render the actual total risk of this portfolio non-zero.

We can easily relax our completeness assumption. Let's assume that our factors do not fully specify all of the covariance among securities. In this case, we have:

$$V_p = \text{SUM}\,[_{i=1 \text{ to } n}\ \text{SUM}\,[_{j=1 \text{ to } n}\ E_{pi}\ E_{pj}\ S_i\ S_j\ R_{ij}\ ]] + \text{SUM}\,[_{k=1 \text{ to } m}\ \text{SUM}\,[_{l=1 \text{ to } m}\ W_k\ W_l\ Z_k\ Z_l\ P_{kl}]] \quad (3)$$

Where Z_k = the standard deviation of residual return for security k
P_{kl} = the correlation between residual returns of security k and security l

It is easy to see that for the special case where $P_{kl} = 0$ for all k not equal to l, then $S_k = Z_k$ and equations (1) and (3) are equivalent.

So we now have one possible solution. If we have a portfolio optimization algorithm that can operate on equation (3) and we include the composite asset in the estimation of the factor model, we have a solution. Unfortunately, few optimization algorithms can operate on equation (3). In addition, it is generally impossible to appropriately include composite assets in the estimation of the factor model. Most composite assets are based on security portfolios where the weights of the included securities vary over time.

Another approach would be to build an optimization algorithm that would look "through" the composite asset to the underlying securities. We would then not need to include the composite asset in the estimation of the factor model. This approach is also problematic. The difficulty here arises if the same security appears in the problem both independently and as a member of a composite asset. Let's imagine a case where our benchmark index is the S&P 500 index. Our portfolio consists of a composite asset for the S&P 100 index, plus fifty other securities including GE stock. GE is a member of the S&P 100 index. Even if our optimization algorithm can look through the S&P 100 composite asset to the underlying securities, another problem arises. When we trade the composite asset, we are really trading each of the 100 member securities, subject to the constraint that each member security be traded in fixed proportions relative to each of the other ninety-nine members. When we trade those same securities as independent securities, no such constraint exists. As such, our algorithm must be able to differentiate GE inside the composite, from GE outside the composite, while knowing that both of these things are equivalent to the GE in the benchmark. This can be accomplished by carefully defining distinct securities (called GE1, GE2 and GE3) under equation (3). This approach cannot be used with equation (1). Unfortunately, we must now have an optimization algorithm that can handle hundreds or thousands of additional constraints as well as deal with the complexities of equation (3).

In addition to portfolios whose members include index contracts, there are many situations where composite assets arise. For example, a large pension plan sponsor may wish to optimize the allocation of monies being managed by independent active managers. It would be valuable if composite assets representing the current portfolio of each manager could be constructed and optimized accordingly.

The Method of Implied Covariance Matrices

Our proposed solution to these difficulties is to transform the problem to the equivalent full-covariance matrix problem (i.e., Markowitz). The elements of the covariance matrix used are implied from the linear factor model.

In a full covariance representation of an optimization problem each asset exists independently, including our composite asset. However, we must specify the variance of each asset and the correlation of each asset to every other asset. These values are readily implied from our linear factor model and current composition of the composite asset. This is not to suggest simply using the observed variances and covariances from the estimation sample. We are computing the set of security variances and correlations that are equivalent to our linear factor model. Such values will be significantly different from the historically observed values. The implied values will also effectively reflect the separation of common factor and security-specific risk that was the reason for our adoption of the factor model.

Equation (1) may be used to compute the variance of each asset. The implied correlation coefficient of between any two assets may be computed as:

$$Q_{ab} = (V_a+V_b-(TE_{ab}\hat{}2)) / (2*S_a*S_b) \qquad (4)$$

Where

Q_{ab} = implied correlation between asset A and asset B
V_a = variance of asset A
S_a = (V_a)^.5 = standard deviation of asset A
TE_{ab} = standard deviation of a portfolio of asset A and minus asset B

The tracking error value TE can also be computed using equation (1). For the case of independent securities A and B, we assume that security A has weight one and security B has weight negative one. For cases involving a composite asset C and other security B, equation (1) is employed with the underlying proportional membership of the composite C included and the contra security B is included at weight negative one. Note that if security B is a member of composite asset C, its active weight is appropriately included (as the sum of the minus one and the positive weight of security B in composite C). For the correlation of two composite assets, C and D, the two proportional memberships of underlying securities are included, using positive weights for composite asset C and negative weights for composite asset D.

Once the asset variances and cross-correlations are computed in this fashion, the optimization problem may be solved using the full covariance matrix. This approach may be implemented using any common portfolio optimization algorithm of sufficient capacity (such as that offered commercially by the author's firm). The computations needed to solve equations (1) and (4) are readily automated using modern spreadsheet software.

Conclusion

The proposed method of using implied covariance matrices allows composite assets to conveniently be included in portfolio optimization problems. The method may be applied to composite assets that represent portfolios of underlying securities such as contracts on stock market indices. As this technique does not require the composite asset to be included in the factor model estimation process, it may be readily used for problems where the proportions of the underlying securities within the composite asset change frequently.

References

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