Portfolio optimization in an uncertain world

Generalized Modern Portfolio Theory

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In this speech we

1. generalize the Markowitz optimization problem

2. solve it

3. link it to risk parity and similar methods
Modern Portfolio Theory

Markowitz’ (1952) mean-variance portfolio optimization problem

- Asset returns ($R$) follow a multivariate normal distribution

\[ R \sim N(\mu, V) \]

- Objective is to maximize portfolio utility ($U_x$) defined on first two moments

\[ U_x = x^T \mu - \lambda \cdot x^T V x \]

- Underlying hypothesis: parameters $\mu$, $V$ are known with certainty.
Generalize Modern Portfolio Theory

We relax the certainty hypothesis,
asset returns don’t strictly obey to predefined laws

accord the definition of utility,
investors seek to minimize loss due to unfavourable price shocks, which can be
foreseen – within the sphere defined by the Hessian
unforeseen – unframed

and accord the optimization objective
minimize loss due to
foreseeable shocks – via variance minimization
unforeseeable shocks – via diversification maximization
Diversification

Diversification: stake, participation, footprint, span, ...

- Concrete measures of diversification
  - Rao’s (1982) squared entropy: \(-x^T x\)
  - Shannon (information theory): \(-x^T \ln(x)\)
  - Roncalli (2013): \(-e^T \ln(x)\)

- Measures that depend on risk estimates
  - Meucci (2009): Eigenvectors of \(V\)
  - Choueifaty & Coignard (2008): \(x^T \sigma / \sigma_p\)
Generalized Modern Portfolio Theory

Generalized portfolio optimization

- We define a level of order $\theta$ (entropy)

  order - multivariate normal $R \sim N(\mu, V_{\theta})$

  disorder - uniform distribution $R \sim U[0,1]$

- Maximize utility

  with return objective $U_x = x^T \mu - \lambda \cdot x^T V_{\theta} x - \theta \cdot e^T \ln(x)$

  without return objective $U_x = -x^T V_{\theta} x - \theta \cdot e^T \ln(x)$
Solution to the generalized problem

Without return objective

\[ \text{Min. } x^T V_\theta x + \theta \cdot e^T \ln(x) \]

\[ \delta x^T V x \quad = \quad \frac{\delta e^T \ln(x)}{\delta x} \quad \iff \quad V x = \frac{1}{x} \quad \iff \quad (V x) \cdot (x) \text{ equal} \]

- Risk parity is optimal !!

Proof by Roncalli (2013)
Risk parity is optimal
in an uncertain world

- By admitting uncertainty one acknowledges the need to protect against the unknown.

- Portfolio is optimal, i.e. loss is minimized, when portfolio variance and diversification optimized.

- Risk parity when variance-over-diversification at maximum.

- Risk parity solution not unique.
  
  frontier of solutions depending on $\theta$ entropy (disorder/uncertainty) 
  several measures of diversification
Optimal solutions depending on $\theta$

- High entropy – optimal portfolio: $1/N$
- Intermediate – optimal portfolio: $1/\text{risk contribution}$
- Low entropy – optimal portfolio: Markowitz
Choice of diversification measure

If Rao’s measure adopted

- Min. $x^T V_\theta x + \theta \cdot x^T x \iff \text{Min. } x^T (V_\theta + \theta \cdot I) x$

- $(V_\theta + \theta \cdot I)$ variance increased with respect to covariance

- This is what happens in
  
  Ledoit & Wolf’s (2003) covariance shrinkage method
  Kritzman’s (2016) stability-adjustment method

- Covariance shrinkage and stability adjusting are optimal !!
Choice of diversification measure

If Kullback-Leibler (cross entropy) measure adopted

- Min. $\sum_i p_i \ln(p_i/q_i)$ s.t. performance target

- If $q = (1/N, \ldots, 1/N)$ the measure is equivalent to Shannon

- Method introduced by


- Entropy-based methods optimal!!
Bayesian optimization is optimal


- Min. $x^T V_B x$

$V_B$ is estimated such that the loss is minimized due to sub-optimality that would arise if the estimates turn out to be wrong.

- A less circumvent formulation:

  Loss is minimized which is partly foreseeable, partly not.

- Bayesian portfolio optimization is optimal !!
Diversification measure based on risk estimates

Diversification Ratio

![Graph showing asset volatility vs. portfolio volatility]

If estimates are wrong, diversification is wrong and inefficient as a result.

- Maximum diversification is not in the same list
Recap

**entropy** $\theta$ intermediate $\theta$ extreme

**without return objective**

*optimal strategies*

- risk parity
- covariance shrinkage
- stability adjusting
- Bayes

**with return objective**

*investment settings*

Strategic asset allocation

- parity approach

high-conviction investing

- Markowitz
- Black-Litterman
Investment example - *equity-bond allocation problem*

**without return objective**

Let $\sigma_{eq} = 15\%$, $\sigma_{bo} = 5\%$, $\rho = 0.2$  

- **risk parity**  
  
  $\begin{bmatrix} \sigma_{eq} \\ \sigma_{bo} \end{bmatrix} = \begin{bmatrix} 25 \\ 75 \end{bmatrix}$  
  
  $x_i = 1/\sigma_i$

- **Markowitz**  
  
  $\begin{bmatrix} \sigma_{eq} \\ \sigma_{bo} \end{bmatrix} = \begin{bmatrix} 5 \\ 95 \end{bmatrix}$  
  
  $x_i \approx 1/\sigma_i^2$

**with return objective**

Let $\mu_{eq} = 6\%$, $\mu_{bo} = 2\%$

- **risk parity**  
  
  $\begin{bmatrix} \sigma_{eq} \\ \sigma_{bo} \end{bmatrix} = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$

- **Markowitz**  
  
  $\begin{bmatrix} \sigma_{eq} \\ \sigma_{bo} \end{bmatrix} = \begin{bmatrix} 25 \\ 75 \end{bmatrix}$

Example put forward by Qian (2011)
1. Top-down approach* (not full optimization)
As covariance more predictable within asset classes than between, Markowitz optimization is deployed within- and risk budgeting between.

2. Bridgewater® All Weather Funds
Four plausible economic scenarios are weighted by their respective volatility levels.

3. Global Macro
Asset class weights set inversely proportional to volatility levels

* Discussed by Scherer (2007)
A bit of philosophy

- Main critic on MPT: "Markowitz optimization vulnerable to estimation error."
  This statement is a misconception. The flaw is the certainty hypothesis.

- Generalized MPT \( U_x = x^T \mu - \lambda \cdot x^T V_\theta x - \theta \cdot e^T \ln(x) \)

  \( \lambda \) risk aversion is a subjective parameter
  \( \theta \) entropy is an objective parameter

- A risk model that defines order differs from a risk model that defines beliefs.
  i.e. \( V_\theta \neq V_0 \)
Risk model that defines order

- Whether to adopt a risk factor, assess if it more damaging to ignore it – so that risk efficiency will be foregone to adopt it – so that diversification will be foregone

Statistical significance may not be an effective criterion.

- We explore two models

  Model 1. one notch down from darkness
  only volatility levels predictable

  Model 2. two notches down from darkness:
  Capital Asset Pricing Model
Risk model that defines order

Model 1. Volatility levels ($\sigma$) predictable

optimal portfolio $x_i = \frac{1}{\sigma_i}$

Model 2. CAPM – market betas ($\beta$) and specific volatilities ($\sigma$) predictable

optimal portfolio* $x_i \propto \frac{1}{\sigma_i}$ and $x_i \propto \frac{1}{\beta_i}$

Model 2\textsuperscript{bis}. constrained CAPM*

$\forall i \in \text{industry } j: \quad R_{it} = \beta_j \cdot R^M_t + \varepsilon_{it}$

optimal portfolio when $x_i = \frac{1}{\sigma_i}$ between individual assets

$x_j = \frac{1}{\beta_j}$ between industry groups

* Clarke et al. (2013) derive the near closed-form solution
Back-test Model \(2^{\text{bis}}\) on corporate bonds

Testbed
- Region: Eurozone
- Period: 2007 to 2016
- Industries: Barclays level 3 sectors aggregated to ten groups
  - Universe: Barclays Euro Corporate
  - Monthly data frequency

Test setup
- Portfolio rebalanced at regular intervals (no foresight)
- \textit{Ex post} performance over test period compared with standard benchmark
Parity portfolio

Ten sectors contribute equally to overall portfolio risk (DTS)
Within sectors firms (bond issuers) contribute equally

As of June 2016
Back-test: return

Parity portfolio in line with benchmark and resilient during crises.

Performance calculations based on monthly total returns data provided by Barclays.
## Key figures

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Parity Index</th>
<th>Parity Portfolio</th>
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<tbody>
<tr>
<td>return</td>
<td>4.8%</td>
<td>5.4%</td>
<td>5.4%</td>
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<tr>
<td>volatility</td>
<td>4.1%</td>
<td>3.4%</td>
<td>3.4%</td>
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<td>Sharpe ratio</td>
<td>1.2</td>
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<td>max drawdown</td>
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<td>-4.3%</td>
<td>-4.1%</td>
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<td>TE</td>
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<td># bonds</td>
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<td>1843</td>
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<td># issuers</td>
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<td>&lt;127</td>
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<td>turnover per month</td>
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<td>10</td>
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<tr>
<td>weight</td>
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<td>-</td>
<td>1.6%</td>
</tr>
</tbody>
</table>

All figures are annual if not specified

Generalized MPT - de Jong
Convex risk profile

Risk parity amortizes drawdowns

QQ plot market vs parity

As of June 2016
Induced quality bias

The parity index has a structural bias towards low-indebted firms
As measured by two debt ratios.
Induced size bias

The parity index has a structural bias towards small firms as measured by two firm fundamentals, sales revenues and book value.
Conclusion

Incorporating uncertainty
and consequently
generalizing the Markowitz optimization problem makes sense.

It corresponds to investment practice,
with a list a portfolio construction methods
with alternative philosophies, e.g. regret theory

Incorporating uncertainty doesn’t suit active high-conviction investment.