Northfield’s 29th Annual Research Conference
March 21, 2017

Investment Horizon and Portfolio Selection

Confidential. Not for further distribution.
Main results

• An objective function that addresses

  *What do you need and when do you need it?*

• Horizon matters, i.e. your portfolio depends explicitly on investment horizon
  - *If you care about shortfall more than you care about surplus*
  - *Even if returns are iid*

• Tactical becomes strategic
Investment Horizon and Portfolio Selection

Martin Tarlie, 2016

https://ssrn.com/abstract=2854336
Agenda

1. Basic idea
2. Operationalizing the *what*
3. Operationalizing the *when*
4. Asset allocation example
5. Two period binomial model example
6. Risk aversion

*What do you need and when do you need it?*
Why does investment horizon matter for your portfolio?

• Natural question

  *When do you need your money?*

• An eternal asset allocation question

  *Are stocks more attractive in the long run?*
### Horizon sensitivity – conventional paradigm

**When does your portfolio depend on horizon?**

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<th>Constant Relative Risk Aversion</th>
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### Horizon sensitivity

**When does your portfolio depend on horizon?**

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<th>Piecewise Power Utility + Asymmetric Preferences</th>
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Where on the efficient frontier?

How much do you care about the variability of your portfolio (or wealth)?

\[
\mu x' - \frac{A}{2} x' \Sigma x
\]
Extending mean variance

Investment risk is not having what you need when you need it

• Focusing on the *needs* and *circumstances* of the investor leads to an expanded set of questions

  1. *What* do you need/desire?
  2. How much do you care about not achieving your need/desire?
  3. *What* do you have?
  4. *When* do you need/desire it?

*What do you need and when do you need it?*
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*What do you need and when do you need it?*
Investment risk is not having what you need when you need it

- Focusing on the *needs* and *circumstances* of the investor leads to an expanded set of questions

1. *What* do you need/desire?
   - Introduce a wealth target
2. How much do you care about not achieving your need/desire?
   - Asymmetric preferences to shortfall and surplus
3. *What* do you have?
4. *When* do you need/desire it?

*What do you need and when do you need it?*
Wealth target + piecewise power law utility

\[ W^* = \text{wealth target} \]
Operationalizing the *what* – the algebra

Wealth target + piecewise power law utility

\[
\text{Shortfall} = \frac{1 - A_\phi}{|1 - A_\phi|} \left( \frac{W}{W^*} \right)^{1-A_\phi} - 1
\]

\[
\text{Surplus} = \frac{1 - A_\pi}{|1 - A_\pi|} \left( \frac{W}{W^*} \right)^{1-A_\pi} - 1
\]

- Asymmetry between attitude to shortfall and surplus is the key driver of horizon sensitivity
- Risk aversion at the target is a measure of this asymmetry
Power law utility

\[ U(W) = \frac{1 - A}{|1 - A|} W^{1-A} \]

- Workhorse utility in financial economics
- Risk aversion (Arrow-Pratt, relative)
  \[ A = -\frac{1}{W} \frac{U''}{U'} \]
  - Risk averse: \( A > 0 \)
  - Risk seeking: \( A < 0 \)
  - Risk neutral: \( A = 0 \)
  - Note: \( W = \) wealth, \( A \neq 1, U' = \partial U / \partial W \)
Power law utility and mean variance

Mean variance ~ power law utility

• Mean variance objective

\[ \mu x' - \frac{A}{2} x'\Sigma x \]

• For power law utility and lognormally distributed wealth

\[ E[W^{1-A}] = e^{(1-A)(\ln E[W] - \frac{A}{2} \text{Var} (\ln W))} \]

This expression follows from

\[ E[W^{1-A}] = e^{(1-A)E[\ln W] + \frac{(1-A)^2}{2} \text{Var} (\ln W)} \]

and using the fact that

\[ \ln E[W] = E[\ln W] + \frac{\text{Var}(\ln W)}{2} \]
Focus on expected shortfall, analogous results for expected surplus

- Expected shortfall utility

\[
\Phi = \int_0^{W^*} \left\{ \frac{1 - A_\Phi}{1 - A_\Phi} \left( \left( \frac{W}{W^*} \right)^{1-A_\Phi} - 1 \right) \right\} P(W) \, dW
\]

Utility of shortfall \hspace{2cm} Probability of shortfall

- The probability weighted sum of the utility of shortfall for all values of wealth less than the target
- Wealth is lognormally distributed, so

\[
P(W) = e^{-\frac{(\ln W - E[\ln W])^2}{2\text{Var}(\ln W)}}
\]

\[
\frac{1}{W \sqrt{2\pi \text{Var}(\ln W)}}
\]
Expected utility – shortfall plus surplus

- Explicit objective function (see Appendix and working paper for details)

\[
\alpha_\phi \Phi\left(E[\ln W], \operatorname{Var}(\ln W); W^*, A_\phi\right) + \alpha_\pi \Pi\left(E[\ln W], \operatorname{Var}(\ln W); W^*, A_\pi\right)
\]

- Mean \quad Variance

Depend on how you invest \quad Investor preferences

- Five preference parameters
  - \( W^* \) = target wealth
  - \( A_\phi \) risk aversion below the target
  - \( A_\pi \) risk aversion above the target
  - \( \alpha_\phi \) and \( \alpha_\pi \) are the weights on expected utility of shortfall and surplus

- Same ingredients as mean-variance objective function
  - Mean variance has a single risk aversion parameter
Interpretation of shortfall

- Expected shortfall accounts for probability and magnitude of shortfall

\[ \Phi \sim - P(W < W^*) + E \left[ \left( \frac{W}{W^*} \right)^{1-A\phi} \right] P \left( W < W^* e^{-(1-A\phi)Var(\ln W)^2} \right) \]

\[ \text{Probability of shortfall} \quad \text{Captures magnitude of shortfall} \]

- This objective function nests the pure probability of shortfall objective

\[ \lim_{A\phi \to -\infty} \Phi \sim - P(W < W^*) \]
A general framework

Piecewise power law + asymmetric preferences

• An objective function that extends the mean-variance paradigm

• Applies to both absolute and relative problems

• Think balance sheet: wealth = assets – liabilities

• Liabilities define the benchmark
  – Consumption in retirement (think target date fund)
  – Plan sponsor with a policy benchmark
  – Pension plan with liabilities
  – S&P 500 for a benchmarked equity manager

• Natural extensions
  – Multiple wealth targets
  – Range of horizons, rather than a single point in time
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*What do you need and when do you need it?*
Operationalizing the *when* – iid example

To operationalize the *when*, need a model of asset returns

- (Log) asset returns are normally distributed – tied to our assumption of lognormally distributed wealth

\[
d \ln R_i(t) = r_i(t) \, dt + \sigma_{Ri} dB_i^R(t)
\]

Expected return \quad Unexpected return

- Wealth dynamics

\[
W(t + dt) = \sum_i \{W(t)x_i(t)\} e^{dt \ln R_i(t)}
\]

Amount invested in asset *i* \quad Return on asset *i*

\(dB_i^R(t)\) is a standard Wiener increment, and \(\sum_i x_i(t) = 1\)
Visualizing the *when*

*Surplus*  

*Shortfall*

*What you need*

*When you need it*
Target compounding rate

Once we introduce time, the target compounding rate emerges

- Target compounding rate (TCR)

\[ W^*(T) = W(0)e^{TCR \cdot T} \]

*What you need*  *What you have*  *When you need it*
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*What do you need and when do you need it?*
Utility function – pure shortfall

\[ \alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0 \]

- Pain of shortfall increases linearly with deviation from target
- Risk neutral below target
- No credit for surplus
Asset returns – iid

- Constant expected returns
  \[ d \ln R_i(t) = \bar{r}_i \, dt + \sigma_{Ri} dB_i^R(t) \]
  Constant expected return

- Stock and bond characteristics

<table>
<thead>
<tr>
<th></th>
<th>Stocks</th>
<th>Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected real return ((\bar{r}_i))</td>
<td>6%</td>
<td>2.5%</td>
</tr>
<tr>
<td>Volatility ((\sigma_{Ri}))</td>
<td>17.5%</td>
<td>7.3%</td>
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</table>

- Assume return correlation equals zero
Objective function

Two ingredients – mean and variance – depend on investment choices

- Objective function

\[ \Phi \left( E \left[ \ln W(T) \right], \text{Var} \left( \ln W(T) \right) \right) ; \text{TCR, } A_\phi \]

Depend on how you invest

Investor preferences

- Mean and variance depend on the weights \( x_i(t) \) invested in asset \( i \) at time \( t \)

- Explicit formulas for mean and variance for iid returns

\[
\text{Var} \left( \ln W(T) \right) = \int_0^T dt \, x'(t) \Sigma x(t)
\]

\[
E \left[ \ln W(T) \right] = \ln W(0) + \int_0^T dt \, \mu' x(t) - \frac{1}{2} \text{Var} \left( \ln W(T) \right)
\]

Note: \( \mu_i = \bar{r}_i + \frac{1}{2} \sigma_i^2 \) is the expected arithmetic return on asset \( i \)
Optimization

Two ingredients – mean and variance – depend on investment choices

- Objective function $\Phi$ depends on the term structure of portfolios (i.e. $x_i(t)$) over the entire investment horizon

- **Directly** optimizing $\Phi$ generates a term structure of optimal portfolios $x_i^*(t)$ over the entire investment horizon
  - The portfolio that matters today is the “current” (i.e. $t = 0$) portfolio $x^*(0)$
  - The optimal portfolios for $t > 0$ reflect the stochastic evolution of wealth (c.f. two period binomial model)
An example – stock-bond allocation

Optimize pure shortfall utility function: $\alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0$

Stock weights $x^+(0)$ as a function of target compounding rate and horizon

<table>
<thead>
<tr>
<th>Target Compounding Rate</th>
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Reward and *investment* risk

Constant target compounding rate, vary horizon

TCR = 5%

Increasing risk

Increasing reward

**Graph:**

- Expected shortfall ($\Phi$) on the x-axis.
- $E[\ln W] - \ln W(0)$ on the y-axis.

Points on the graph:
- $T = 1$: stock wgt=28%
- $T = 5$: stock wgt=46%
- $T = 10$: stock wgt=59%
- $T = 15$: stock wgt=67%
- $T = 20$: stock wgt=74%
Reward and *investment* risk

Constant horizon, vary target compounding rate (TCR)

\[ T = 10 \text{ years} \]

- Increasing risk
- Increasing reward

<table>
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<tr>
<th>TCR</th>
<th>Stock weight</th>
<th>Expected wealth shortfall</th>
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<tr>
<td>3%</td>
<td>41%</td>
<td>0.44</td>
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Expected wealth shortfall:
- 0.4
- 0.42
- 0.44
- 0.46
- 0.48
- 0.5
- 0.52
- 0.54
- 0.56
- 0.58
Compare to pure power law (aka mean variance)

Optimize pure shortfall utility function: \( \alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0 \)

Stock weights as a function of target compounding rate and horizon

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Power utility weights

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Assume return correlation of zero between stocks and bonds
Where on the efficient frontier?

What’s your target compounding rate and horizon?

![Graph showing the efficient frontier with volatility on the x-axis and expected geometric return on the y-axis. Points labeled A = 2, A = 3, and A = 6 are indicated on the graph.](image)
Where on the efficient frontier?

What’s your target compounding rate and horizon?

![Graph showing the efficient frontier with points labeled A=2, A=3, A=6, T=10, T=20, TCR=5%](image-url)
Where on the efficient frontier?

What’s your target compounding rate and horizon?

![Graph showing the efficient frontier with points labeled A = 2, A = 3, A = 6, T = 10, T = 20, and T = 3, with corresponding expected returns and volatilities.](image)
Where on the efficient frontier?

What’s your target compounding rate and horizon?

![Graph showing the efficient frontier with specific points labeled for different time horizons and expected returns.](image-url)
Expected shortfall is “mostly” M-V efficient

\( \Phi(x, y) \) is a universal function of \( x = \text{Var}(\ln W) \) and \( y = E[\ln W] - \ln W^* \)

\[ \frac{\partial \Phi}{\partial y} > 0 \]

\[ \frac{\partial \Phi}{\partial x} < 0 \]

\[ \frac{\partial \Phi}{\partial x} > 0 \]
An example – stock-bond allocation

Optimize pure shortfall utility function: $\alpha_\phi = 1, \alpha_\pi = 0, A_\phi = 0$

Stock weights as a function of target compounding rate and horizon

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Sensitivity to expected returns – iid case

TCR: target compounding rate
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*What do you need and when do you need it?*
A simple example

• Contrast target compounding rate and target wealth

• Tactical becomes strategic

• Optimization
Two period binomial model

A simple example

- Two assets
  - A store of wealth with zero return, zero volatility
  - A volatile asset with 6% expected real return, and 17% annualized vol

\[ r = \text{binomial return} \]

\[ p_u = 60\% \]
\[ r_u = +20\% \]

\[ p_d = 40\% \]
\[ r_d = -15\% \]
Risk aversion at the target is a measure of how much you care about not achieving the target.

Piecewise power law:

\[ \alpha_\phi = 1.0, A_\phi = 2.7 \]
\[ \alpha_\pi = 0.3, A_\pi = 5.0 \]
Two period binomial model

A simple example to illustrate constant TCR vs. constant wealth target

• Two periods
  – Start with $W_0 = 1$, target $W^* = 1.13$
  – Invest for two periods
    \[
    W_1 = W_0(1 + r_0 x_0) = 1 + r_0 x_0
    \]
    \[
    W_2 = W_1(1 + r_1 x_1) = (1 + r_0 x_0)(1 + r_1 x_1)
    \]

• How much to invest in the volatile asset at $t = 0$?
  – Maximize expected utility at $t = 2$
    \[
    \max_{x_0, x_1} V(x_0, x_1)
    \]
    \[
    V(x_0, x_1) = p_up_u U(W_2^{uu}) + p_up_d U(W_2^{ud}) + p_dp_u U(W_2^{du}) + p_dp_d U(W_2^{dd})
    \]
  – where $W_2^{uu} = (1 + r_u x_0)(1 + r_u x_1)$, $W_2^{ud} = (1 + r_u x_0)(1 + r_d x_1)$, ...
Directly optimizing the two period objective generates a term structure of portfolios.

- $x_{0|0}^*$: optimal weight in the volatile asset at $t = 0$, conditional on information at $t = 0$
  - $x_{0|0}^* = 50\%$

- $x_{1|0}^*$: optimal weight in the volatile asset at $t = 1$, conditional on information at $t = 0$
  - $x_{1|0}^* = 50\%$
Two period binomial model

$x_{1|0}^*$ accounts for the stochastic dynamics of wealth

$W_1^u = 1.1$

$x_{1|1}^* = 30\%$

$x_{0|0}^* = 50\%$

$x_{1|0}^* = 50\%$

$W_1^d = 0.93$

$x_{1|1}^* = 75\%$
Relationship to dynamic programming

Dynamic programming and the direct approach give the same answer for $x_{0|0}^*$

- Basic problem

$$\max_{x_0, x_1} E_0[U(W_2)]$$

$$W_2 = \frac{W_0(1 + r_0x_0)(1 + r_1x_1)}{W_1}$$

- Rewrite the basic problem as

$$\max_{x_0} E_0 \left[ \max_{x_1} E_1[U(W_2)] \right]$$

- Dynamic programming works backwards, by

1. first solving $J_1(W_1) = \max_{x_1} E_1[U(W_2)]$ as a function of $W_1$
2. then solving $J_0(W_0) = \max_{x_0} E_0[J_1(W_1)]$ as a function of $W_0$

- Both the direct method and dynamic programming give the same answer for the optimal "current" weight $x_{0|0}^*$ at $t = 0$, conditional on information at $t = 0$
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What do you need and when do you need it?
How much do you care about not achieving your target?

- Asymmetric preferences between shortfall and surplus reflect how much you care about not achieving your target
  - Can we characterize this asymmetry?
How much you care about not achieving the target?

Risk aversion at the target is a measure of how much you care

\[
\text{Risk aversion } = A_\phi \\
\frac{1 - A_\phi}{|1 - A_\phi|} \left( \left( \frac{W}{W^*} \right)^{1 - A_\phi} - 1 \right)
\]

\[
\text{Risk aversion } = A_\pi \\
\frac{1 - A_\pi}{|1 - A_\pi|} \left( \left( \frac{W}{W^*} \right)^{1 - A_\pi} - 1 \right)
\]
Risk premium and risk aversion

How much are you willing to pay to avoid a fair gamble?

• When wealth is either **below** or **above** the target
  
risk premium \( \sim A_2 \times \text{variance of the gamble} \)

\[
A_2 = \begin{cases} 
A_\phi, & \text{if } W < W^* \\
A_\pi, & \text{if } W > W^* 
\end{cases}
\]
is the coefficient of **second order** risk aversion

• When wealth is **at** the target
  
risk premium \( \sim A_1 \times \text{std dev of the gamble} \)

– \( A_1 \) is the coefficient of **first order** risk aversion

– When you care about shortfall more than surplus, i.e. \( \alpha_\phi |1 - A_\phi| > \alpha_\pi |1 - A_\pi| \)

\[
A_1 = 1 - \frac{\alpha_\pi |1 - A_\pi|}{\alpha_\phi |1 - A_\phi|}
\]
First order risk aversion

Is a rough measure of how much you care about not achieving the target

\[ A_1 = 1 - \frac{\alpha_\pi |1-A_\pi|}{\alpha_\phi |1-A_\phi|} \text{ if } \alpha_\phi |1-A_\phi| > \alpha_\pi |1-A_\pi| \]

Shortfall matters more than surplus

<table>
<thead>
<tr>
<th>Pure power law</th>
<th>(\alpha)</th>
<th>(A)</th>
<th>(\alpha)</th>
<th>(A)</th>
<th>(A_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binomial model example</td>
<td>1</td>
<td>2.7</td>
<td>0.3</td>
<td>5</td>
<td>0.29</td>
</tr>
</tbody>
</table>
Extensions

- Mean reverting expected returns (see SSRN paper)
- Shortfall over a range of times rather than a single point
- Multiple wealth targets
Summary

Investment risk is not having what you need when you need it

Specifying

1. Target wealth or target compounding rate
2. Investment horizon
3. Preferences for shortfall and surplus

leads to a framework that addresses

1. *What* do you need/desire?
2. How much do you care about not achieving your need/desire?
3. *What* do you have?
4. *When* do you need/desire it?

_horizon comes to the fore_
Expected utility – shortfall

Objective function depends on “mean”, “variance”, and preferences

- Key point

\[
\Phi \left( E[\ln W], \text{Var}(\ln W); W^*, A_\phi \right)
\]

Expected log wealth Variance of log wealth

Depend on how you invest Investor preferences

- The explicit formula for expected shortfall utility is

\[
\Phi = \frac{1 - A_\phi}{|1 - A_\phi|} \left\{ -N(z_1) + e^{(1-A_\phi)(E[\ln W] - \ln W^*) + \frac{1}{2}(1-A_\phi)^2 \text{Var}(\ln W)} \right. \\
\left. \times N\left( z_2(A_\phi) \right) \right\}
\]

- Definitions
  - \( N(\cdot) = \) standard cumulative normal
  - \( z_1 = \frac{\ln W^* - E[\ln W]}{\sqrt{\text{Var}(\ln W)}} \), \( z_2(A_\phi) = z_1 - (1 - A_\phi)\sqrt{\text{Var}(\ln W)} \)

- Resembles price of a European put option for stock price = strike price = 1
  - \( P = -N(-d_1) + e^{-rt} N(-d_2) \)
  - But results from straightforward evaluation of the expectation integral, there are no replicating portfolios or no arbitrage assumptions
Multiple target compounding rates

$$\mathbb{E} [\ln W] - \ln W(0)$$

Expected shortfall ($\Phi$)

- TCR = 4%
- TCR = 5%
- TCR = 6%

$T = 20$
$T = 15$
$T = 10$
$T = 5$
Reward and *investment* risk

Vary target compounding rates

![Graph showing the relationship between expected wealth shortfall and compounding rates.](image)

- TCR = 3%
- TCR = 5%
- TCR = 7%
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