Risk contributions: duality and sensitivity

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Section 1: Introduction

This section discusses the origins of the concept of a risk contribution in investment management.

It also presents four distinct ways of defining risk contributions.

The geometric way of looking at them provides several insights.
Motivation and goals

- *Risk contributions* were introduced into the world of finance by Fischer Black and Robert Litterman in the 80’s and 90’s.
- Risk contributions were adopted wholeheartedly by practitioners in investment management after Litterman (1996).
- The concept got little attention from academics who focus on investments.
- The goal of this talk is to present three little-known facts about risk contributions.
Origins of risk contributions

- Fischer Black and Robert Litterman (1993) gave a central role to risk contributions in a course for portfolio managers that they co-taught at Goldman Sachs, as discussed in Litterman (1996).
More recent work related to this talk

1.2 Four approaches to the definition of risk contributions

The fact that there are four quite distinct ways to describe risk contributions is an indication of their importance.
The starting point is always a \textit{portfolio decomposition}

\[ P = P_1 + P_2 + P_3 \]

expressing a portfolio $P$ as a sum of other portfolios $P_i$.

The $P_i$s may be holdings of individual assets, factors, idiosyncratic risks, and so forth.

The goal is to attribute the risk of $P$ to the components $P_1, P_2,$ and $P_3$ in an additive way.
Four approaches to the definition of risk contributions

1. Beta × portfolio risk
2. Column sums of a covariance matrix
3. Euler's theorem for homogeneous functions, applied to portfolio volatility (≡ standard deviation)
4. Orthogonal projection, which is a geometric interpretation of the first approach
Approach 1: Beta × portfolio risk

So given a portfolio decomposition \( P = P_1 + P_2 + P_3 \) we want the risk contributions \( C_1, C_2, \) and \( C_3 \) to attribute the risk of \( P \) to the components \( P_1, P_2, \) and \( P_3 \) in an additive way, so that:

\[
\sigma_P = C_1 + C_2 + C_3.
\]

The standard definition of the risk contributions is

\[
C_i \equiv \frac{\text{Cov}(P_i, P)}{\sigma_P} = \beta_i \sigma_P
\]

where \( \beta_i = \text{Cov}(P_i, P)/\sigma_P^2 \) is the regression coefficient in the regression of \( P_i \) on \( P \).
Approach 2: Column sums interpretation

Portfolio decomposition:

\[ P = P_1 + P_2 + P_3. \]

The covariance matrix of the components \( P_i \):

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}.
\]

The variance contributions \( \mathcal{V}_i \) are defined as the column sums and the risk contributions \( \mathcal{C}_i \) are those divided by \( \text{STD}(P) \):

\[ \mathcal{V}_i \equiv \sigma_{1i} + \sigma_{2i} + \sigma_{3i} \quad \text{and} \quad \mathcal{C}_i \equiv \frac{\mathcal{V}_i}{\text{STD}(P)} \quad \text{for} \ i = 1, 2, 3. \]
Approach 3: Euler’s theorem I

Portfolio volatility is a homogeneous function $\sigma(w) \equiv \sqrt{w^T \Sigma w}$ of portfolio weights $w$...

and so its tangent planes go through the origin (see next slide)...

and so portfolio volatility is the sum of the weights times the marginal contributions to risk...

$$\sigma(w) = \sum_i w_i \frac{\partial \sigma(w)}{\partial w_i}.$$  

The terms on the right are the contributions to risk. (The partial derivatives are the marginal contributions to risk.)
Approach 3: Euler’s theorem II

Figure showing that the tangent plane to a homogeneous function (whose graph must be a cone) goes through the origin.
Approach 4: Orthogonal projection interpretation

If we project the components of the portfolio decomposition $P = P_1 + P_2 + P_3$ orthogonally onto $P$ itself, we get the risk contributions $C = (C_1, C_2, C_3)$. The thin lines show the directions along which the orthogonal projection moves the line segments representing the components $P_1$, $P_2$, and $P_3$ of $P$. 
Section 2: Three new facts about risk contributions

Looking at risk contributions geometrically leads to these insights:

1. For a given portfolio decomposition, the risk contributions and risks of the components must satisfy certain conditions.

2. For a given portfolio decomposition, there is an alternative decomposition — the “dual decomposition” — that may provide additional insights. See Grinold (2011).

3. When either the risk contributions or risks of the components are large, the portfolio may be sensitive to risk regime changes.
Three new facts about risk contributions

Fact 1: The range of possible risk contributions and risks of a portfolio decomposition is given explicitly. The key condition may be called the “bracelet condition”. (Analog: the range of all covariance matrices is the set of positive semidefinite matrices.)

Fact 2: We show that the “dual portfolio decomposition” gives the same risk contributions and, in a sense to be explained, no other such decomposition exists.

Fact 3: We show that sensitivity of portfolio risk to a risk regime change is governed substantially by the risk contributions and risks of the components of a decomposition.
2.2 The range of possible risk contributions and risks

A covariance matrix is a summary of the risk of some random quantities.

Similarly, the risk contributions and risks summarize the risk of the components of a decomposition.

Just as covariance matrices are characterized as positive semidefinite matrices, there is also a simple characterization of the risk contributions and risks of the components of a decomposition.
Understanding the range of risk contributions and risks

In a portfolio decomposition, the risk contributions \( C_i \) of the components are central and are almost always reported in buy-side risk reports.

A key role is also played by the risks

\[
S_i \equiv \sqrt{\sigma_{ii}} = \text{STD}(P_i)
\]

of the components \( P_i \) of the decomposition. These are also often presented in risk reports.

We want conditions on two sets of numbers \( c_i \) and \( s_i \) that ensure that they are the risk contributions \( C_i \) and risks \( S_i \) of some portfolio decomposition.
The residual risks $\mathcal{T}_i$

We will have use for the **residual risks** $\mathcal{T}_i$ of the components, defined by

$$\mathcal{T}_i^2 = S_i^2 - \beta_i^2 \sigma_P^2 = S_i^2 - C_i^2.$$  

$\mathcal{T}_i^2$ is the residual variance in the (ex ante) regression of $P_i$ on $P$.

Note that the variance of a component is the sum of the squares of its risk contribution and its residual risk:

$$S_i^2 = C_i^2 + \mathcal{T}_i^2.$$
A trick for financial fraud investigators

Suppose a TAA strategy is composed of three components: stocks, bonds, and currencies. The following risk contributions and risks are impossible! (Note that the table presents $C_i, S_i, T_i, i = 1, 2, 3$.)

<table>
<thead>
<tr>
<th>Component contributions</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>2.0%</td>
</tr>
<tr>
<td>Bonds</td>
<td>4.0%</td>
</tr>
<tr>
<td>Currencies</td>
<td>3.0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Risks</th>
<th>Risks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stocks</td>
<td>8.0%</td>
</tr>
<tr>
<td>Bonds</td>
<td>5.0%</td>
</tr>
<tr>
<td>Currencies</td>
<td>5.0%</td>
</tr>
</tbody>
</table>
A trick for financial fraud investigators

Suppose a TAA strategy is composed of three components: stocks, bonds, and currencies. The following risk contributions and risks are impossible! (Note that the table presents $C_i, S_i, T_i, i = 1, 2, 3$.)

![Risk contributions and risks](image)
The residual risks $T_i$ must form a bracelet!

The following are the only conditions that risk contributions and risks must always satisfy, unless portfolio risk is zero.

- Clearly the $C_i$'s must sum to a positive number.
- Clearly $S_i \geq C_i$.
- The only remaining condition is that the residual risks satisfy

$$T_i \leq \sum_{j \neq i} T_j \text{ for all } i.$$

The last is the condition that it is possible to make a bracelet from links of length $T_i$. No individual link can be longer than the sum of all the others.
Graphic of the bracelet condition

Each link represents a component of a decomposition.

The lengths are the residual risks.

If the longest link is longer than the sum of all the other links, then it is impossible to close the gap and make a bracelet.

Currencies (4% idiosyncratic risk)

Stocks (7.7%)

Bonds (3%)
A note on dimension

Given putative risk contributions $c_i$ and risks $s_i$ that satisfy the necessary and sufficient conditions, it is always possible to construct a portfolio decomposition having these risk contributions and risks *in no more than 3 dimensions* – i.e., with only three sources of randomness.

The reason is that if you put an $n$-dimensional bracelet on a table it will lie flat. A bracelet can always be arranged to fit in a 2-dimensional space.

Thus you can never infer intrinsically 4-dimensional information from a report of risk contributions and risks. You can from a covariance matrix.
2.3 Duality

Grinold (2011) recognized the duality between decomposing a portfolio in terms of either characteristic portfolios or factor mimicking portfolios.

Here we give a slight extension of that idea and show that the two decompositions always produce the same risk contributions.
Dual decompositions

Dual decompositions are one way of describing Grinold’s (2011) SX and MY decompositions (characteristic portfolios versus factor-mimicking portfolios).

Start with the risk contributions as orthogonal projections:

\[ P = P_1 + P_2 \quad \text{and} \quad \sigma_P \equiv \text{STD}(P) = C_1 + C_2. \]
Dual decompositions

Define a new decomposition $P = Q_1 + Q_2$ using the “dual basis” $Q_1, Q_2$ satisfying $Q_1 \perp P_2$ and $Q_2 \perp P_1$: 
Dual decompositions give the same risk contributions

The new decomposition $P = Q_1 + Q_2$ produces the same risk contributions because the perpendiculatrs are coincident at the orthocenter (known to Archimedes but not to Euclid!).
2.4 Sensitivity

If risk contributions and risks are large compared to portfolio risk, then the portfolio is the result of large but offsetting effects.

This hints that large risk contributions and risks may flag sensitivity of portfolio volatility to a change in the risk regime.

Such a change may cause hedges to become less effective or even to break down, and portfolio volatility may spike.

This subsection provides a mathematical quantification of this simple insight.
2.3 Risk model perturbations we consider

Perturb volatilities according to

$$S_i' = S_i(1 + \delta_i)$$

and write $R'$ for the perturbed correlations.

Then we will use

$$||\delta|| = \sqrt{\sum \delta_i^2}$$

as our measure of the size of the perturbation to the volatilities and

$$||R' - R|| = \text{maximum eigenvalue or spectral radius of } R' - R$$

as our measure of the size of the perturbation to the correlations.
What the sensitivity depends on

Then the impact on portfolio variance is bounded in terms of $\|\delta\|$ and $\|R' - R\|$, as well as these descriptors of the decomposition:

- $\phi \equiv \sum S_i / \sigma$, the diversification ratio of the portfolio, which is the factor by which portfolio risk $\sigma$ would increase if “correlations among the components went to 1”.

- $H(S) \equiv \sum S_i^2 / (\sum S_i)^2$, the Herfindahl measure of concentration of the component risks $S$.

- $H(C) \equiv \sum C_i^2 / (\sum C_i)^2$, the Herfindahl measure of concentration of the risk contributions $C$. 
Sensitivity of portfolio risk: the bound

The relative perturbation in the portfolio volatility is bounded as follows:

$$\frac{\left|\sigma_P'^2 - \sigma_P^2\right|}{\sigma_P^2} \leq \phi^2 H(S) \| R' - R \| + 2\sqrt{H(C)} \| \delta \| + \text{higher order terms}$$

This bound is seen to be larger if
- the diversification ratio $\phi$ is higher,
- the risks $S$ are more concentrated, and
- the risk contributions $C$ are more concentrated.

This supports Litterman’s (1996) practice of presenting position sizes, which give a good indication of the risks $S$, as well as risk contributions $C$ in “Hot Spots” reports.
Section 3: More on duality

Using the orthocenter theorem to prove that the dual decomposition gives the same risk contributions works only in 2 dimensions. We will take a look at higher dimensions.

It turns out that under a certain condition ("irreducibility") on the covariance matrix of the components that there is only one alternative decomposition that gives the same risk contributions.

Do dual decompositions arise naturally? Indeed they do.

If we decompose a portfolio into factor components and an uncorrelated residual, then the decomposition is reducible because all the correlations between the residual and the other components are zero. So the previous result doesn't apply.

However, there is a good resolutions to this.
3.1 Generalizing the orthocenter idea

Algebra rather than geometry allows us to handle higher dimensions.

By formulating the problem as one of invariance of risk contributions under a change of basis, we get not only a generalization to any dimension but also a uniqueness property of the dual decomposition.

The key condition for uniqueness is called *irreducibility*, which means that the covariance matrix of the components not be block diagonal.
Dual decompositions give the same risk contributions

We need to turn this geometry into algebra. Write $X$ for a basis and $h$ for the holdings or coordinates of a generic portfolio $P$:

$$P = Xh = \sum_{i=1}^{K} h_i X_i.$$  

With $\Omega$ denoting the covariance matrix of the basis $X$, the variance contributions $\mathcal{V}$ of this decomposition are

$$\mathcal{V} = \text{diag}(h)\Omega h.$$  

We want to see how this behaves under a change of basis.
A change of basis for a decomposition

Define a new basis $\tilde{X}$ in terms of an invertible matrix $M$, and the corresponding new holdings $\tilde{h}$:

$$\tilde{X} \equiv XM \text{ and } \tilde{h} \equiv M^{-1}h.$$

Then of course

$$\tilde{X}\tilde{h} = XMM^{-1}h = Xh = P$$

so the portfolios $\tilde{P}_i \equiv \tilde{h}_i\tilde{X}_i$ also form a decomposition of $P$. 
The condition for invariance under a change of basis

Now the new variance contributions are

$$\text{diag}(\tilde{h})\tilde{\Omega}\tilde{h} = \text{diag}(M^{-1}h)M^T\Omega MM^{-1}h$$

$$= \text{diag}(M^{-1}h)M^T\Omega h$$

and invariance under the change of basis means

$$\text{diag}(h)\Omega h = \text{diag}(M^{-1}h)M^T\Omega h \text{ for all } h \in \mathbb{R}^K.$$
Dual decompositions give the same risk contributions

If we choose $M = \Omega^{-1}$, then new basis is indeed a dual basis:

$$\text{Cov}(\tilde{X}, X) = \Omega^{-1}\Omega = I.$$ 

The resulting decomposition

$$\tilde{X} = X\Omega^{-1} \quad \text{and} \quad \tilde{h} = \Omega h.$$ 

is called the dual decomposition.

**Theorem:** The dual decompositions give the same risk contributions as the original decomposition.

**Proof:**

$$\text{diag}(\tilde{h})\tilde{\Omega}\tilde{h} = \text{diag}(\Omega h)\Omega^{-1}\Omega h = \text{diag}(\Omega h)h = \text{diag}(h)\Omega h.$$
Let's call a covariance matrix *irreducible* if it is not block diagonal.

**Theorem (uniqueness of the dual in the irreducible case):** If the covariance matrix $\Omega$ of the basis $X$ is irreducible, then the dual basis is the only change of basis (up to a scaling) that always produces the same risk contributions.
3.2 Constraint attribution and dual decompositions

If dual decompositions didn’t arise naturally, they would not be of much interest.

They arise naturally in considering constraint attribution in portfolio optimization.

The also arise as the factor mimicking portfolios associated with a given set of factor portfolios as in Grinold (2011).
Do dual decompositions occur nature? Yes.

A constrained optimal portfolio may be expressed as a linear function of the Lagrange multipliers of the constraints. This leads to a risk attribution relative to an associated basis.

It may also be expressed as a linear function of the right hand sides of the constraints. This leads to a risk attribution relative to another basis.

These two bases are mutually dual!

Therefore they give the same risk contributions.

(But they do not give the same risks.)
3.3 The reducible case also arises naturally

It often happens that the irreducibility condition isn’t satisfied.

For example, suppose we are decomposing a portfolio into components explained by several factors plus an idiosyncratic residual.

Then the decomposition is *reducible* because all the correlations between the residual and the other components are zero. So the previous result doesn’t apply.

However, there is a good resolutions to this.
The duality theorem in the reducible case

Assume that the positive definite covariance $\Omega$ of the basis $X$ has been relabeled to put it in the block-diagonal form $(\Omega_{ij})$ where the diagonal blocks $\Omega_{qq}, q = 1, 2, \ldots, p$, are irreducible positive definite matrices. Then the change of basis $M$ preserves risk contributions if and only if $M$ is of the form

$$M = \begin{pmatrix} \Omega_{11}^{\gamma_1} & 0 & \cdots & 0 \\ 0 & \Omega_{22}^{\gamma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega_{pp}^{\gamma_p} \end{pmatrix} D,$$

where $\gamma_q \in \{0, 1\}, q = 1, 2, \ldots, p$, and $D$ is a $K \times K$ invertible diagonal matrix.
The number of decomposition mappings that produce the same risk contributions

Under the same assumptions, there are precisely $2^r$ distinct decomposition mappings giving the same risk contributions, where $r$ is the number of diagonal blocks of $\Omega$ having more than one element.
A commonplace reducible portfolio decomposition

While this is an abstract result, it implies that in a factor decomposition

\[ r_P = \sum_{i=1}^{k} \beta_i f_i + \epsilon \]

where

- \( r_P \) is the portfolio return;
- \( f_i \) is the \( i \)th of \( k \) factors;
- \( \beta_i \) is the portfolio's exposure to factor \( f_i \); and
- \( \epsilon \) is a residual uncorrelated with the factors.

The previous theorem again implies that there is a unique dual decomposition, as long as the factor covariance matrix itself is irreducible.
An illustration

The first table presents volatilities and correlations of four assets.

The chart gives the holdings of a portfolio of these assets.

<table>
<thead>
<tr>
<th>ASSET CORRELATIONS AND VOLS</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSET</td>
<td>A1</td>
<td>A2</td>
<td>A3</td>
<td>A4</td>
</tr>
<tr>
<td>A1</td>
<td>1.0</td>
<td>0.9</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>A2</td>
<td>0.9</td>
<td>1.0</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>A3</td>
<td>0.6</td>
<td>0.6</td>
<td>1.0</td>
<td>0.7</td>
</tr>
<tr>
<td>A4</td>
<td>0.5</td>
<td>0.5</td>
<td>0.7</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Volatility: 16% 16% 13% 10%

Portfolio Weights

<table>
<thead>
<tr>
<th></th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50%</td>
<td>25%</td>
<td>50%</td>
<td>25%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>-25%</td>
<td>-50%</td>
<td>-50%</td>
</tr>
</tbody>
</table>

Risk contributions: duality and sensitivity
Asset-based risk attribution

This is a basic risk attribution giving risk contributions and risks of the four assets.

The dual risks are the risks from the dual decomposition. They flag sensitivity to expected returns.
Factor and residual attribution example

Here we flesh out the risk model above by giving it in factor form.

<table>
<thead>
<tr>
<th>FACTOR CORRELATIONS AND VOLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
</tr>
<tr>
<td>F1</td>
</tr>
<tr>
<td>F2</td>
</tr>
<tr>
<td>Volatility</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>FACTOR EXPOSURES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor/Asset</td>
</tr>
<tr>
<td>F1</td>
</tr>
<tr>
<td>F2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>RESIDUAL VOLATILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Residual</td>
</tr>
<tr>
<td>Volatility</td>
</tr>
</tbody>
</table>
Factor and residual attribution example

The portfolio may be thought of as a sum of factor exposures and residuals.

The factor-based decomposition provides a different view. The dual is still unique despite reducibility.

![Risk attribution to factors and residuals](image)
A set of risk contributions and risks is valid iff certain conditions hold, in particular the bracelet condition that no component’s residual risk exceeds the sum of all the other residual risks.

For every portfolio decomposition there is a natural dual decomposition giving the same risk contributions. Moreover, this is the only change of basis with that property in the irreducible case.

In the reducible case we can characterize all such changes of basis.

Large risk contributions or risks are harbingers of sensitivity of portfolio risk to a risk regime change.

Although risk contributions have been around for more than two decades, these facts appear not to have been well known.
Thank you!

QSCR 18239 (August 2018)
Brief bibliography


Brief bibliography (continued)

