Analytical Solutions of Optimal Portfolio Rebalancing

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Summary

- The optimal portfolio rebalancing problem
- Related research
- Motivation for this study
- New results from this study
  - Published in Quantitative Finance 2018

The Optimal Portfolio Rebalancing Problem

- The goal is to track a target portfolio closely without paying excessive transactions costs
- Ad-hoc heuristics are often used in practice, such as monthly rebalancing or setting fixed bands around the target weights, but they tend to be too simplistic
- The problem can be modeled as seeking the optimal trade-off between tracking error to the target and the transactions costs incurred, similar to the mean-variance analysis

$$\min \frac{1}{2k}(x - x_T)^T V (x - x_T) + \left| x - x_C \right|^T \cdot tc$$

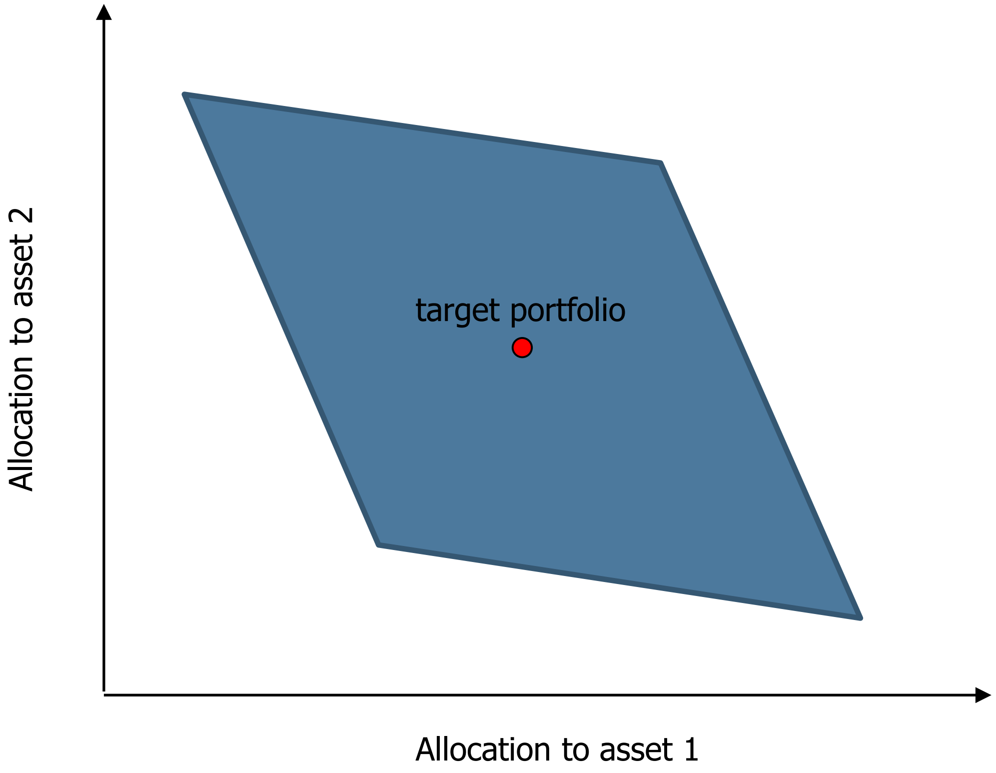
- x_T vector of target portfolio weights
- x_C vector of current portfolio weights
- tc vector of proportional transactions costs
- V asset by asset covariance matrix
- k investor's risk aversion parameter towards tracking error

Previous Studies

- Most related
  - Dybvig, P.H., Mean-variance portfolio rebalancing with transaction costs. 2005.
  - Donohue, C. and Yip, K., Optimal portfolio rebalancing with transaction costs: Improving on calendar- or volatility-based strategies. 2003.
  - Mei, X., DeMiguel, V. and Nogales, Francisco J., Multiperiod portfolio optimization with multiple risky assets and general transaction costs. 2016.
  - Masters, Seth J., Rebalancing: Establishing a consistent framework. 2003.
- Continuous time model
  - Leland, Hayne E., Optimal portfolio management with transactions costs and capital gains taxes. 1999.
  - Liu, H., Optimal consumption and investment with transaction costs and multiple risky assets. 2004.
  - Dumas, B. and Luciano, E., An exact solution to a dynamic portfolio choice problem under transaction costs. 1991.
  - Davis, M.H.A. and Norman, A.R., Portfolio selection with transaction costs. 1990.

The No-Trade Region

- Optimal rebalancing strategy defines a no-trade region around the target portfolio
- If current portfolio is in the region, no trading occurs; if current portfolio is not in the region, trading occurs to move it to the boundary of the region, not back to the target portfolio
- We derive the equation of the no-trade region



Motivation for This Study

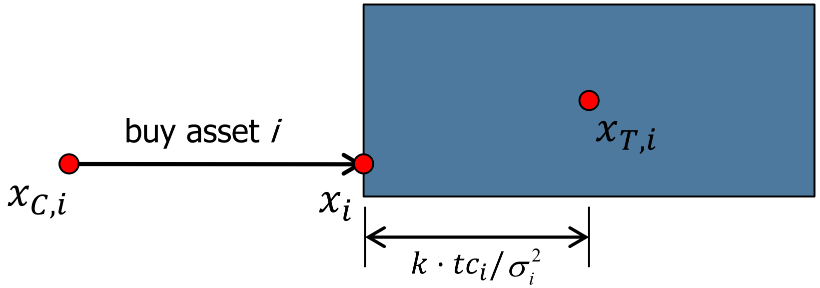
- Except for the trivial case of a single asset, no previous studies have provided any analytical solutions of the optimal portfolio
- Transactions costs are non-linear and there are no general analytical solutions for optimal portfolio rebalancing with multiple assets, but …
- In the book "Modern portfolio theory and investment analysis" by Elton, Gruber, Brown and Goetzmann, a chapter is written on "simple techniques for determining the efficient frontier"
- Use simple rules to determine the mean-variance optimal portfolio with simplifying models of the covariance matrix: the single-index model and the constant-correlation model
- These simplifying models increase the accuracy with which covariances can be forecast
- Rank assets based on simple formulas and set cutoff values: make it very clear why an asset does or does not enter the optimal portfolio
- Are similar results possible for optimal portfolio rebalancing? The answer is yes!

New Results

- The first set of analytical solutions and conditions of optimal portfolio rebalancing with simplifying models of the covariance matrix
- Uncorrelated returns
  - Funds of hedge funds often seek to allocate to strategies that are at least approximately uncorrelated
- Constant-correlation model
  - Outperforms other techniques when used to estimate future correlations from historical data (Elton and Gruber 1973)
- One-factor model
  - Securities co-move only because of their exposures to a common factor, e.g. the market
- With and without the riskless asset
- Theoretical interest rather than practical value
  - New insights of the optimal portfolio, interesting connection with computational geometry

Uncorrelated Returns with the Riskless Asset

$$x_{C,i} \begin{cases} < x_{T,i} - k \cdot tc_i / \sigma_i^2 & \text{for buy assets} \\ \in [x_{T,i} - k \cdot tc_i / \sigma_i^2, x_{T,i} + k \cdot tc_i / \sigma_i^2] & \text{for no - trade assets} \\ > x_{T,i} + k \cdot tc_i / \sigma_i^2 & \text{for sell assets} \end{cases}$$



$$x_i = \begin{cases} x_{T,i} - k \cdot tc_i / \sigma_i^2 & \text{for buy assets} \\ x_{C,i} & \text{for no - trade assets} \\ x_{T,i} + k \cdot tc_i / \sigma_i^2 & \text{for sell assets} \end{cases}$$

- Always rebalance to the boundary of the no-trade region
- Never rebalance away from target: only true for uncorrelated returns with the riskless asset

Uncorrelated Returns with the Riskless Asset (Example)

Asset	Volatility	T-Costs	Target Weights	Current Weights	$x_{T,i} - \frac{k \cdot tc_i}{\sigma_i^2}$	$x_{T,i} + \frac{k \cdot tc_i}{\sigma_i^2}$	Action	Optimal Weights
1	5%	0.01%	10%	5%	8.0%	12.0%	Buy	8.0%
2	10%	0.20%	10%	15%	0.0%	20.0%	No-Trade	15.0%
3	15%	0.30%	10%	5%	3.3%	16.7%	No-Trade	5.0%
4	20%	0.30%	10%	15%	6.3%	13.8%	Sell	13.8%
5	25%	0.01%	10%	5%	9.9%	10.1%	Buy	9.9%
6	5%	0.40%	10%	15%	-70.0%	90.0%	No-Trade	15.0%
7	10%	0.50%	10%	5%	-15.0%	35.0%	No-Trade	5.0%
8	15%	0.50%	10%	15%	-1.1%	21.1%	No-Trade	15.0%
9	20%	0.60%	10%	5%	2.5%	17.5%	No-Trade	5.0%
10	25%	0.60%	10%	15%	5.2%	14.8%	Sell	14.8%

- $k = 0.5$

Uncorrelated Returns without the Riskless Asset

$$\frac{1}{k}\sigma_i^2(x_{C,i}-x_{T,i})+tc_i<\lambda_0 \quad \text{for buy assets}$$

$$\frac{1}{k}\sigma_i^2(x_{C,i}-x_{T,i})-tc_i>\lambda_0 \quad \text{for sell assets}$$

$$x_i = \begin{cases} x_{T,i}+k\cdot(\lambda_0-tc_i)/\sigma_i^2 & \text{for buy assets} \\ x_{C,i} & \text{for no - trade assets} \\ x_{T,i}+k\cdot(\lambda_0+tc_i)/\sigma_i^2 & \text{for sell assets} \end{cases}$$

- Need to determine λ_0 but only need to check it against the *2n* "critical numbers"
- When λ_0 falls in between two adjacent critical numbers, the buy/sell/no-trade assets are fixed regardless of the value of λ_0
- Sort the critical numbers and loop through all the intervals (including the endpoints)
- For each interval, back-out λ_0 assuming it falls on this interval and using the condition that all weights sum up to 1. Check if its value is valid with the assumption
- Keep all valid solutions and select the one with the smallest objective function value

Uncorrelated Returns without the Riskless Asset (Example)

		Assuming λ_0 equals the current critical number				Assuming λ_0 falls between the current and the next critical number			
Critical Number	Asset i	Update to *B/S/N*	λ_0	Equals critical number?	Objective Function	Update to *B/S/N*	λ_0	Falls in expected range?	Objective Function
-1.00%	9	9 from S to N							
-0.64%	5	5 from S to N							
-0.62%	5					5 from N to B	-0.24%	No	
-0.60%	7	7 from S to N	-0.21%	No			-0.21%	No	
-0.53%	3	3 from S to N	-0.19%	No			-0.19%	No	
-0.38%	6	6 from S to N	-0.07%	No			-0.07%	No	
-0.28%	8	8 from S to N	-0.06%	No			-0.06%	No	
-0.10%	2	2 from S to N	-0.05%	No			-0.05%	Yes	0.05%
-0.04%	1	1 from S to N	-0.12%	No			-0.12%	No	
-0.02%	1		-0.12%	No		1 from N to B	-0.03%	No	
0.02%	10	10 from S to N	-0.03%	No			-0.03%	No	
0.08%	3		-0.03%	No		3 from N to B	-0.02%	No	
0.10%	4	4 from S to N							
0.20%	9					9 from N to B			
0.30%	2					2 from N to B			
0.40%	7					7 from N to B			
0.43%	6					6 from N to B			
0.70%	4					4 from N to B			
0.73%	8					8 from N to B			
1.23%	10					10 from N to B			

Uncorrelated Returns without the Riskless Asset (Result)

Asset	Volatility	T-Costs	Target Weights	Current Weights	Action	Optimal Weights
1	5%	0.01%	10%	5%	Sell	2.84%
2	10%	0.20%	10%	15%	No-Trade	15.00%
3	15%	0.30%	10%	5%	No-Trade	5.00%
4	20%	0.30%	10%	15%	Sell	13.18%
5	25%	0.01%	10%	5%	Buy	9.55%
6	5%	0.40%	10%	15%	No-Trade	15.00%
7	10%	0.50%	10%	5%	No-Trade	5.00%
8	15%	0.50%	10%	15%	No-Trade	15.00%
9	20%	0.60%	10%	5%	No-Trade	5.00%
10	25%	0.60%	10%	15%	Sell	14.43%

- It is optimal to sell asset 1 from 5% to 2.84%, even though it is targeted at 10%
- Asset 1 is used to adjust the total weights of the optimal portfolio given its small transactions cost and low volatility

The Constant-Correlation Model

- Let ρ be the common correlation among all assets. There exists a cutoff value C such that:

$$C \begin{cases} > (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i} & \text{for buy assets} \\ \in [(x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i}, (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i}] & \text{for no - trade assets} \\ < (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i} & \text{for sell assets} \end{cases}$$

$$x_i = \begin{cases} x_{T,i} + C/\sigma_i + k(\lambda_0 - tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for buy assets} \\ x_{C,i} & \text{for no - trade assets} \\ x_{T,i} + C/\sigma_i + k(\lambda_0 + tc_i)/((1-\rho)\cdot\sigma_i^2) & \text{for sell assets} \end{cases}$$

- If $\rho = 0$ then $C = 0$ and the problem reduces to the case of uncorrelated returns

The Constant-Correlation Model with the Riskless Asset

- Use the same algorithm as uncorrelated returns without the riskless asset to search for *C*, but with a different set of critical numbers
- Same set of assets as in previous examples, but with pairwise correlation 0.5

Asset	Volatility	T-Costs	Target Weights	Current Weights
1	5%	0.01%	10%	5%
2	10%	0.20%	10%	15%
3	15%	0.30%	10%	5%
4	20%	0.30%	10%	15%
5	25%	0.01%	10%	5%
6	5%	0.40%	10%	15%
7	10%	0.50%	10%	5%
8	15%	0.50%	10%	15%
9	20%	0.60%	10%	5%
10	25%	0.60%	10%	15%

The Constant-Correlation Model with the Riskless Asset

		Assuming C equals the current critical number				Assuming C falls between the current and the next critical number			
Critical Number	Asset i	Update to $B/S/N$	C	Equals critical number?	Objective Function	Update to $B/S/N$	C	Falls in expected range?	Objective Function
-7.75%	6	6 from S to N	-1.97%	No			-1.97%	No	
-5.50%	7	7 from S to N	-1.58%	No			-1.58%	No	
-4.00%	9	9 from S to N	-1.28%	No			-1.28%	No	
-2.75%	3	3 from S to N	-1.07%	No			-1.07%	No	
-2.58%	8	8 from S to N	-0.82%	No			-0.82%	No	
-1.50%	2	2 from S to N	-0.68%	No			-0.68%	No	
-1.29%	5	5 from S to N	-0.53%	No			-0.53%	No	
-1.21%	5		-0.53%	No		5 from N to B	-0.66%	No	
-1.15%	10	10 from S to N	-0.54%	No			-0.54%	Yes	0.03%
-0.50%	4	4 from S to N	-0.55%	No			-0.55%	No	
-0.45%	1	1 from S to N	-0.61%	No			-0.61%	No	
-0.05%	1		-0.61%	No		1 from N to B	-0.42%	No	
1.25%	3		-0.42%	No		3 from N to B	0.00%	No	
2.00%	9		0.00%	No		9 from N to B	0.40%	No	
2.50%	4		0.40%	No		4 from N to B	0.75%	No	
2.50%	2		0.75%	No		2 from N to B	1.00%	No	
3.65%	10		1.00%	No		10 from N to B	1.33%	No	
4.08%	8		1.33%	No		8 from N to B	1.64%	No	
4.50%	7		1.64%	No		7 from N to B	1.92%	No	
8.25%	6		1.92%	No		6 from N to B	2.50%	No	

The Constant-Correlation Model with the Riskless Asset

Asset	Volatility	T-Costs	Target Weights	Current Weights	Action	Optimal Weights
1	5%	0.01%	10%	5%	Sell	3.20%
2	10%	0.20%	10%	15%	No-Trade	15.00%
3	15%	0.30%	10%	5%	No-Trade	5.00%
4	20%	0.30%	10%	15%	Sell	14.80%
5	25%	0.01%	10%	5%	Buy	7.68%
6	5%	0.40%	10%	15%	No-Trade	15.00%
7	10%	0.50%	10%	5%	No-Trade	5.00%
8	15%	0.50%	10%	15%	No-Trade	15.00%
9	20%	0.60%	10%	5%	No-Trade	5.00%
10	25%	0.60%	10%	15%	No-Trade	15.00%

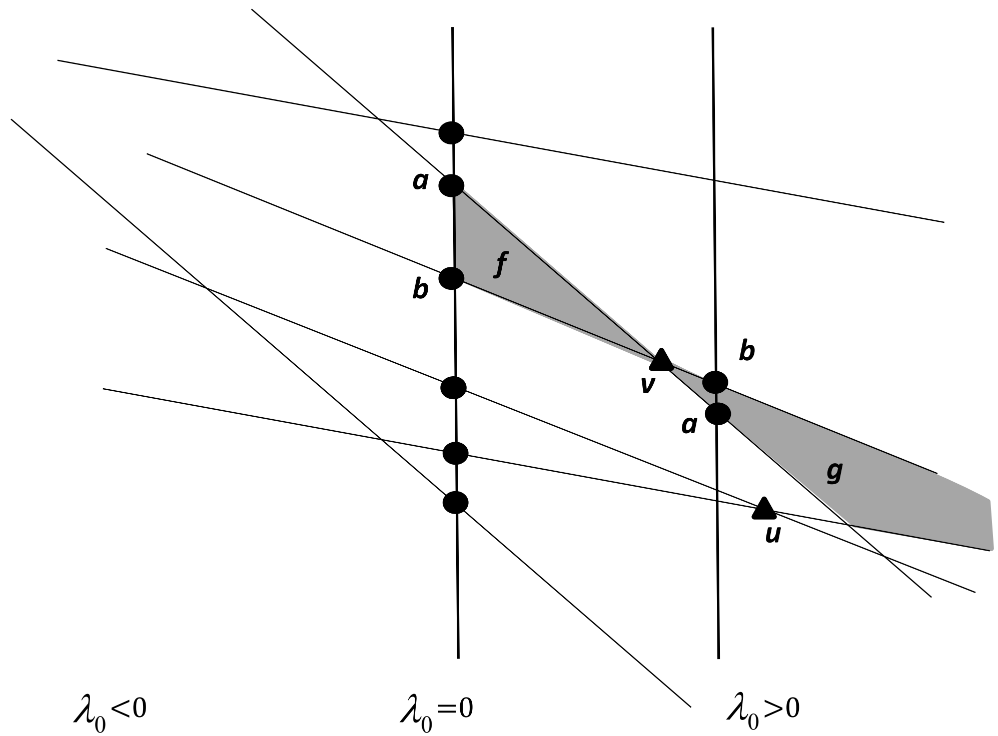
- It is optimal to sell asset 1 from 5% to 3.20%, even though it is targeted at 10%
- Same phenomenon as in the case of uncorrelated returns without the riskless asset, but for a different reason …
- Correlation effect: underweighting asset 1 offsets the tracking error from overweighting assets 8 and 10 given the positive correlation

The Constant-Correlation Model without the Riskless Asset

$$C \begin{cases} > (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i} & \text{for buy assets} \\ \in [(x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i}, (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 - tc_i)}{(1-\rho)\sigma_i}] & \text{for no-trade assets} \\ < (x_{C,i} - x_{T,i})\sigma_i - \dfrac{k(\lambda_0 + tc_i)}{(1-\rho)\sigma_i} & \text{for sell assets} \end{cases}$$

- It is now a 2 dimensional problem: need to search for both C and λ_0
- As λ_0 moves from negative infinity to positive infinity, each critical number traces out a line
- All these lines induce a division of the plane into faces, edges and vertices, which are collectively called a *line arrangement*
- Locate the point (λ_0, C) along with the cell of the line arrangement it falls into that give the smallest objective function value among all valid solutions
- Traverse the line arrangement and dynamically update the set of buy/sell/no-trade assets for each face, edge and vertex

A Little Adventure into Computational Geometry



- First determine the buy/sell/no-trade assets of each point and each interval on the line $\lambda_0 = 0$
- Move this line to the right: assign the buy/sell/no-trade assets of ***a*** to ***av***, ***b*** to ***bv***, ***ab*** to ***f***
- As the line hits ***v***, copy (and adjust) the buy/sell/no-trade assets from ***a*** to ***v*** and to ***ba***
- Update the buy/sell/no-trade assets of ***a*** and ***b*** as they cross each other on the vertical line

The One-Factor Model

- σ_M volatility of the common factor
- β_i beta of asset *i* to the common factor
- $\sigma_{\varepsilon,i}$ idiosyncratic risk of asset *i*
- σ_i total volatility of asset *i*

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2$$

- There exists a cutoff value C such that:

$$x_i = \begin{cases} x_{T,i} + C\beta_i / \sigma_{\varepsilon,i}^2 + k(\lambda_0 - tc_i)/\sigma_{\varepsilon,i}^2 & \text{for buy assets} \\ x_{C,i} & \text{for no - trade assets} \\ x_{T,i} + C\beta_i / \sigma_{\varepsilon,i}^2 + k(\lambda_0 + tc_i)/\sigma_{\varepsilon,i}^2 & \text{for sell assets} \end{cases}$$

$$C \begin{cases} > \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i > 0 \\ \in [\dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i}, \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i}] & \text{for no - trade assets with } \beta_i > 0 \\ < \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i > 0 \end{cases}$$

$$C \begin{cases} < \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i} & \text{for buy assets with } \beta_i < 0 \\ \in [\dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 - tc_i)}{\beta_i}, \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i}] & \text{for no - trade assets with } \beta_i < 0 \\ > \dfrac{(x_{C,i} - x_{T,i})\sigma_{\varepsilon,i}^2 - k(\lambda_0 + tc_i)}{\beta_i} & \text{for sell assets with } \beta_i < 0 \end{cases}$$

- If β_i = 0 then reduces to the case of uncorrelated returns

The One-Factor Model with the Riskless Asset (Example)

- Use essentially the same algorithm as uncorrelated returns without the riskless asset to search for *C*, but with a different set of critical numbers and some tweaks to account for the sign of beta

- In the example below, the common factor volatility is 15% and Kelly is 0.5

Asset	Beta	Idiosyncratic Risk	T-Costs	Target Weights	Current Weights	Action	Optimal Weights
1	1.2	5%	0.01%	10%	5%	Sell	-6.19%
2	1.1	10%	0.20%	10%	15%	No-Trade	15.00%
3	1	15%	0.30%	10%	5%	No-Trade	5.00%
4	0.9	20%	0.30%	10%	15%	Sell	12.90%
5	0.8	25%	0.01%	10%	5%	Buy	9.44%
6	1.2	5%	0.40%	10%	15%	No-Trade	15.00%
7	-0.3	10%	0.50%	10%	5%	No-Trade	5.00%
8	1	15%	0.50%	10%	15%	No-Trade	15.00%
9	-0.5	20%	0.60%	10%	5%	No-Trade	5.00%
10	0.8	25%	0.60%	10%	15%	Sell	14.32%

- It is optimal to sell asset 1 from 5% to -6.19%, even though it is targeted at 10%

- Correlation effect: underweighting asset 1 offsets the tracking error from overweighting positive beta assets (6 and 8) and from underweighting negative beta assets (7 and 9)

What Happens If …

- Reduce the idiosyncratic risk of asset 5 from 25% to 5%
- Its optimal weight changes from 9.44% (buy) to 2.57% (sell), while the optimal weight of asset 1 increases to -2.14%

Asset	Beta	Idiosyncratic Risk	T-Costs	Target Weights	Current Weights	Action	Optimal Weights
1	1.2	5%	0.01%	10%	5%	Sell	-2.14%
2	1.1	10%	0.20%	10%	15%	No-Trade	15.00%
3	1	15%	0.30%	10%	5%	No-Trade	5.00%
4	0.9	20%	0.30%	10%	15%	Sell	13.09%
5	0.8	5%	0.01%	10%	5%	Sell	2.57%
6	1.2	5%	0.40%	10%	15%	No-Trade	15.00%
7	-0.3	10%	0.50%	10%	5%	No-Trade	5.00%
8	1	15%	0.50%	10%	15%	No-Trade	15.00%
9	-0.5	20%	0.60%	10%	5%	No-Trade	5.00%
10	0.8	25%	0.60%	10%	15%	Sell	14.42%

What Happens If …

- Also increase the beta of asset 5 to 1.2 to match with asset 1
- The optimal weights of both assets 1 and 5 are 1.24%

Asset	Beta	Idiosyncratic Risk	T-Costs	Target Weights	Current Weights	Action	Optimal Weights
1	1.2	5%	0.01%	10%	5%	Sell	1.24%
2	1.1	10%	0.20%	10%	15%	No-Trade	15.00%
3	1	15%	0.30%	10%	5%	No-Trade	5.00%
4	0.9	20%	0.30%	10%	15%	Sell	13.25%
5	1.2	5%	0.01%	10%	5%	Sell	1.24%
6	1.2	5%	0.40%	10%	15%	No-Trade	15.00%
7	-0.3	10%	0.50%	10%	5%	No-Trade	5.00%
8	1	15%	0.50%	10%	15%	No-Trade	15.00%
9	-0.5	20%	0.60%	10%	5%	No-Trade	5.00%
10	0.8	25%	0.60%	10%	15%	Sell	14.51%

What Happens If …

- Further reduce the idiosyncratic risk of asset 5 from 5% to 3%
- Its optimal weight becomes -3.06% while it is optimal not to trade asset 1

Asset	Beta	Idiosyncratic Risk	T-Costs	Target Weights	Current Weights	Action	Optimal Weights
1	1.2	5%	0.01%	10%	5%	No-Trade	5.00%
2	1.1	10%	0.20%	10%	15%	No-Trade	15.00%
3	1	15%	0.30%	10%	5%	No-Trade	5.00%
4	0.9	20%	0.30%	10%	15%	Sell	13.44%
5	1.2	3%	0.01%	10%	5%	Sell	-3.06%
6	1.2	5%	0.40%	10%	15%	No-Trade	15.00%
7	-0.3	10%	0.50%	10%	5%	No-Trade	5.00%
8	1	15%	0.50%	10%	15%	No-Trade	15.00%
9	-0.5	20%	0.60%	10%	5%	No-Trade	5.00%
10	0.8	25%	0.60%	10%	15%	Sell	14.62%

Conclusion

- Introduced the optimal portfolio rebalancing problem and the no-trade region concept
- Presented analytical solutions and conditions of the optimal portfolio with simplifying models of the covariance matrix
  - Uncorrelated returns
  - Constant-correlation model
  - One-factor model
  - Reviewed algorithms and examples with and without the riskless asset
  - Insights into the optimal portfolio
- Read the paper for details and proofs

Thank You!