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A Heuristic Approach for Delta Hedging in Discrete Time

Dan diBartolomeo

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Introduction

- One of the most basic concepts in modern finance is that of "delta hedging" of an option.
- It can be shown that under the *assumption of continuous rebalancing*, the value of an option is independent from the expected return of the underlying asset, but remains highly dependent on the expected volatility of the return of the underlying asset.
- The relationship between the percentage price movements of an underlying asset, and the percentage price movements of an option on the underlying assets is generally referred to as the "delta" of the option.
- In mathematical terms it is the first derivative of the option value with respect to the value of the underlying asset.

Motivation

- In actual traded markets, assumption of continuous rebalancing is unrealistic.
- It would only make sense in theory if transaction costs were zero, while real world option trading is generally exposed to relatively high transaction costs.
- At the other end of the time horizon spectrum, the economic payoffs of options at expiration are well understood to typically have skew and other important features even if the distribution of the returns of the underlying asset is normal.
- The question of how one would go about risk assessment and hedging of option positions over discrete time intervals has been less explored in the academic literature.

Literature Review

- Derman (*Risk*, 1999) proposes corrections to the Black Scholes option pricing for the situation of discrete time hedging.

- Sepp (*Journal of Investment Strategies*, 2013) tries to optimize discrete time delta hedging in the presence of known transaction costs.

- Park, Lee and Choe (*East Asian Journal of Applied Mathematics*, 2016) explore the distribution of hedging errors using a recursive process to update delta values.

- Perhaps closest in sprit to this presentation is Peeters, Dert and Lucas (2003, *Tinbergen Institute Working Paper*) who show that traditional hedge ratios are inappropriate for portfolios of options in discrete time.

The Black Scholes Model

- First published 1973 (*Journal of Political Economy*).
- The properties of the Black Scholes (BS) option pricing model are very well understood. As "delta" is the first derivative of option value to underlying value, there are also well defined second and third derivatives, usually known respectively as "gamma" and "speed".
- The derivatives of option values to other inputs such as the volatility of the underlying, the prevalent risk free interest rate and the passage of time toward expiration of the option all have familiar closed form solutions.
- Values for gamma and other "Greeks" generally represent instantaneous derivatives practitioners are often presented with values that are actually implied values assuming daily rather than continuous rebalancing.

Black Scholes in Discrete Time

- When the time horizon for risk assessment and position rebalancing are of material length (e.g. a month), the value of the most appropriate hedge ratio becomes a complex issue.

- It is a function of time decay, the financial leverage between the option and underlying prices, the skew of the option payoffs, and other properties of the problem that are subject to change during the time interval to the horizon.

- All of these different aspects of the option interact in non-linear ways.

Limitations of Black Scholes

- While a cornerstone of modern finance, the Black-Scholes (BS) option model has limitations in the context of a multi-asset class portfolio.

- BS assumes that interest rates are fixed during the life of the option, which clearly an untenable idea for interest rate dependent options.

- Many other option pricing models exist that are appropriate to particular security types, but the issue of how to delta hedge a portfolio of options in discrete time with other option models has not been well addressed in the financial literature.

- The heuristic method presented here is not dependent on the form of the option pricing model, although our empirical example is done with a simple BS model so as to be easily understood and reproduced by the reader.

Our Proposal

- In this method, we make two adjustments to the traditional delta value.

- The first addresses the time decay in the value of the option within the risk horizon and the changes that will be occurring in financial leverage as the respective values of the underlying asset and the related option.

- The second adjustment reflects the “risk effect” of skew in the payoffs at the time the of the risk horizon.

Options and Factor Risk Models

- There is a practical problem of a how to represent a pair of option positions within a larger portfolio using a factor risk model.
- There are a couple possible representations of the problem.
  - If we think of the classical "delta hedge" the most obvious representation is what we call a composite asset (a tradeable portfolio like an ETF), with a "delta neutral" percentage of the underlying adjusted for the leverage implied by the relative prices of the underlying and the option. The balance of the composite asset portfolio is the risk free asset.
  - A second representation is to create a data record where the factor exposures and idiosyncratic risk of the underlying asset are multiplied by: (option delta * leverage). However, care must be taken to recognize that the *idiosyncratic risk of the underlying and the related risk of the option are correlated nearly at one, rather than the usual assumption of zero for "idiosyncratic" risks.*

A Step by Step Illustration

- To illustrate the problem, we will posit an underlying security with price 100, annual volatility 30, and that the current interest rate is 2% per annum. We will have both a call option and a put option with strike price of 105 with 90 calendar days to expiration. As such, the call is "out of the money" and the put is "in the money".

- To make replicating this process easier, I'll use a standard BS calculator that is available free online at https://www.niftytrader.in/option-pricing-calculator/.

- Please note the calculation values are approximate as this online calculator would not accept fractional underlying prices or fractional numbers of trading days.

Start the Exercise with “Button Pushing”

- Our first task is to price the existing option.
- We can obtain all the “Greeks” from the free calculator but for the moment all we need is the theoretical option prices and delta values which are respectively $3.56 (delta = .414) for the call and $8.04 (delta = .590) for the put.
- We know that the if our expectation for volatility for the underlying asset is equal to the market’s expectation of volatility, the option is fairly priced and *the expected return on the option must be close to zero.*
- If our expectation of underlying asset volatility is different from the market’s expectation, the expected return of an option over a finite interval can be obtained from procedure in Rubinstein (*Journal of Finance*, 1984).

Moving to A Discrete Time Horizon

- We are interested in obtaining the best delta or hedge ratio conditional on the time horizon of risk being ten trading days (14.5 calendar days).

- Consider we what expect to happen to the option prices under three scenarios.
  - First, that the price of the underlying asset remains the same at 100. Given that return distribution of the underlying is presumed normal, the mean and mode of the expected return for the underlying will be around zero.
  - Our second scenario is that the underlying price will make a large downward move to the 1% confidence interval return (Z = -2.33).
  - Our third scenario will be the mirror event, where the price is presumed to move upward to the 99% confidence level of the asset return distribution (Z = 2.33).

- We note that given an annual underlying volatility of 30%, the corresponding value for 10 trading days is (30 * (10/252)^.5) or 5.98%.

After Ten Trading Days Pass

- At the end of the 10 trading day horizon, the new price of the call conditional on no price change in the underlying is $2.98, an option return of -16.29%.
  - For the put, the corresponding price at the horizon is $7.54, an option return of -6.22%.
  - *These values represent the modes of the respective option return distributions.*

- To do this we note that at the 1% confidence interval (underlying price down), the call value after 10 trading days is essentially zero for a -100% return. The put price is $11.12 for a return of 38.31%.

- At the 99% confidence interval underlying return (underlying price up), we obtain a price of $9.55 a return of 168.25% and $.12 for the put, a negative return of 98.5%

Volatility of the Horizon Option Returns

- We now have three return values for the call option conditional on returns of the underlying (Z = 0, Z = -2.33, Z = 2.33)

- The two extreme return values are supposed to be 4.66 stand deviations apart implying that the standard deviation of the call option return distribution is 57.56% including the price leverage associated with the ratio of the underlying price and the option price at the starting time.

- Adjusting for leverage ($100/$3.59) we obtain that the 10 day volatility of the unlevered option return distribution is 2.066%.

- If we assumed that leverage was constant and gamma was zero, the expected value of this volatility would just be the original delta of the option times the 10 day volatility of the underlying (.414 * 5.98) or 2.47%.

First Adjustment

- So the "effective delta" inclusive of the impact of gamma and time decay in the option values is (2.066/2.471 * .414) or .345.

- Repeating the same arithmetic steps for the put option yields a value of (2.36/3.52 * -59) or -.395 for revised delta.

- Given both the mean and mode of the option return distributions we can estimate the skew of those distributions as (Mean – Mode) / (standard deviation).
  - The mean values must be zero if the option is fairly priced
  - The mode value is just the option return assuming no change in the price of the underlying.

Second Adjustment: Accounting for Payoff Skew

- With all this information in hand, we can turn to the impact of skew in the payoffs over the finite horizon.

- Given the return values for the mean, mode and standard deviation of the call and put return distributions, the respective values of skew are both positive at .28 and .212 (mode is lower than the mean).

- We can use the Cornish Fisher (1938) method to adjust the delta again for the economic impact of skew (gamma).
  - https://en.wikipedia.org/wiki/Cornish–Fisher_expansion.
  - Basically, we adjust the expected volatility of the option return again to account for skew as compared to the current expectation of option return volatility.

Combined Adjustment

- For our current example of the call option, we now have an expected return = 0, an expected option return volatility = 57.56, and an estimated skew of .28, which yields an effective volatility of 51.73.

- Scaling delta by this relation for the call option (51.73/57.56) yields a final delta for the call option of .31, materially different from the initial value of .414.

- Applying Cornish-Fisher to our put option we obtain an effective volatility of 27.18% for the option return distribution. Scaling the delta accordingly provides (-.395 * 27.18/29.36) or -.366, again materially lower in magnitude than the initial delta of -.59.

Conclusions

- The heuristic presented here is easily implemented for any option model.

- It produces option hedge ratios appropriate to the sort of finite time intervals (e.g. 10 trading days) that are often of practical consequence at the portfolio level and consistent with many regulatory reporting regimes. |

- Given that this process routinely produces lower magnitude delta values, it is consistent with literature such as Froot (NBER, 1993) suggesting that delta hedging "over-hedges".

- Most importantly, it allows for the risk of portfolios containing options to be assessed in a *computationally very efficient fashion as compared to traditional Monte Carlo simulations, as each option need be priced not more than four times* (as opposed to thousands of times).