Intra-Horizon Risk  
By Nick Wade

The concepts of tracking error and value-at-risk (VaR) are widely adopted as risk management measures within the investment community. However, they are not free of criticism. For example, both tracking error and VaR have been criticized widely in the literature for the lack of information they provide about tail risk. One of the key assumptions underlying both tracking error and VaR is that security or portfolio returns are Normally distributed. The distributions of return observed in markets – particularly at higher frequencies such as daily or intra-day – are non-Normal; they exhibit fat tails and often skew. The use of a measure that relies upon the Normal distribution therefore underestimates the size of the possible loss to the extent that the actual distribution deviates from Normal. Various measures have been proposed to resolve that issue.1

The concept of “tracking error” came out of index fund management as a way of quantifying the extent to which a particular fund deviated from an index. In active management, as the differences from a particular benchmark become more and more marked, so the relevance and usefulness of tracking error in its unmodified form as a risk measure decrease. For example, see the recent presentations by diBartolomeo on “Strategy Risk.”2

In addition, reliance on the Normal distribution ignores the effects of jumps in returns or volatility, both of which have increasing importance at shorter horizons.3 In fact, the Basel Committee noted the importance of jumps and their omission from VaR in the Overview to the Amendment to the Capital Accord to Incorporate Market Risk (1996).

However, there is yet another dimension in which both tracking error and VaR are inadequate measures of risk, and that is in the time dimension. Both tracking error and VaR characterize the return distribution at the end of some investment period, and say nothing whatsoever about the return path followed by the investment during the period. Again, this omission was noted by the Basel Committee in the same 1996 document. Typically VaR is presented as a 10-day number, and tracking error as an annual number. However, neither measure tell us the probability or magnitude of the likely loss within that horizon, or “intra”-horizon.4 We will use VaR-I to denote intra-horizon VaR.

The magnitude of intra-horizon risk is important in several situations:

1. Survival Risk: where there is a “floor” we cannot breach and remain in business. This is of particular concern to leveraged investors such as trading desks or hedge funds.

2. Monitored Investors: where the asset owner may remove the mandate due to poor performance within a particular horizon. If short-run performance is poor, the manager may be fired.

3. Valuation of Collateral: for example in securities lending or for capital adequacy regulations

4. Retirement/Endowment: for example, funding a liability intra-horizon

Luckily, this is not a new problem and has been the focus of great attention in the world of barrier-option pricing since active trading in barrier options began back in the 1960’s. Of particular concern with barrier options is the probability of breaching a particular barrier during a holding period. Our concern is clearly related; given a probability level e.g. 1%, what barrier (VaR) might be reached?

To answer this question, we resort to a concept from statistics called “first passage probability,” and also known as first hitting time. This is simply a group of measures that describe the probability that a particular level will be reached, or the expected time it will take to be reached, during a given horizon.

The first-passage probability for the Normal distribution is well known, including instances where there is drift in the mean.5 However, as previously noted, the Normal distribution is not always a suitable assumption. In cases where we need a more flexible distribution, we are forced to look for first passage solutions to more flexible distributions that allow jumps. Some examples would be Merton’s “jump diffusion” model, or the two-sided pure jump model of Carr, Geman, Madan and Yor, or the finite-moment log stable model of Carr and Wu. These models can capture a much richer variety of distributions, but at the cost of some additional effort. There are few analytical solutions in these cases,6 and typically we are forced to solve a partial integro-differential equation numerically by resorting to finite difference methods, binomial or trinomial trees, or Monte Carlo simulation.7

In the case of the Normal distribution without drift, we can derive the first passage probability and an expression for intra-horizon VaR quite easily by exploiting a relationship known as the principle of reflection. This simply states that for a given return path under the Normal distribution, in the absence of drift, there is an equivalent equally probable mirror-image return path.
For example, we can imagine a return path A that descends, breaches a barrier, and then recovers to end the period at some level “A” above the barrier. From the point at which A breaches the barrier we can draw a mirror-image path B that rises above the barrier mirroring A’s fall below it, and then descends below it to land at some point “-A” below the barrier, mirroring A’s rise above. Thinking about the probabilities for a moment, we can see that the path-dependent joint probability of breaching the barrier and ending the period at -A below the barrier. The really fun bit is that the path-dependent joint probability of breaching the barrier and ending the period at A is, by the principle of reflection, equal to the path-dependent joint probability of breaching the barrier and ending the period at -A below the barrier. In the absence of drift, things are symmetric about the barrier.

We can then do a bit of probability maths:

\[
\text{Prob (breach barrier) } = 1 - \text{prob (never breach)}
\]

\[
= 1 - \text{joint prob (end up above, and min always above)}
\]

\[
= 1 - \text{prob (end above) + joint prob (end above, and min} <= \text{barrier)}
\]

\[
= \text{prob (end below) + joint prob (end above, and min} <= \text{barrier)}
\]

We know from our discussion above that the second term in the last line; the joint probability of breaching the barrier and recovering to some level above the barrier is equal to the probability of ending at a symmetric point below.

Following the logic above, the probability of breaching a barrier b within a horizon T becomes:

\[
\text{Pr(breach barrier) } = N \left[ \frac{\ln \left( \frac{B}{S} \right) - \mu T}{\sigma \sqrt{T}} \right] + \left( \frac{B}{S} \right) \cdot \left( \frac{\ln \left( \frac{B}{S} \right) + \mu T}{\sigma \sqrt{T}} \right)
\]

Note that I’ve used Girsanov’s Theorem to convert a Normal distribution into a path-independent one.

We can clearly see that the introduction of drift has had a significant impact on both VaR and intra-horizon VaR, as suggested by the other work mentioned on “strategy risk.” Instead of 10.7% bigger at 1% confidence without drift, VaR-I becomes 19.1% bigger at 10% drift, and 28.5% bigger at 15% drift. The multiplier for wider confidence bands (e.g. 2.5% or 5%) is worse still. For example, at the 5% confidence level and with 15% drift VaR-I is now 1.75 times the size of VaR – almost double.

Moving on, let’s consider the impact of drift on VaR-I – but still with the Normal distribution. For comparative purposes, look at what happens when we introduce drift of 10% and 15% respectively. (see table, below)

<table>
<thead>
<tr>
<th>PROB</th>
<th>µ</th>
<th>VaR</th>
<th>VaR-I</th>
<th>VaR-I/VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>10%</td>
<td>1.053σ√T</td>
<td>1.493σ√T</td>
<td>1.417</td>
</tr>
<tr>
<td>2.5%</td>
<td>10%</td>
<td>1.368σ√T</td>
<td>1.752σ√T</td>
<td>1.281</td>
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<tr>
<td>1%</td>
<td>10%</td>
<td>1.735σ√T</td>
<td>2.067σ√T</td>
<td>1.191</td>
</tr>
<tr>
<td>5%</td>
<td>15%</td>
<td>0.720σ√T</td>
<td>1.262σ√T</td>
<td>1.753</td>
</tr>
<tr>
<td>2.5%</td>
<td>15%</td>
<td>1.035σ√T</td>
<td>1.504σ√T</td>
<td>1.453</td>
</tr>
<tr>
<td>1%</td>
<td>15%</td>
<td>1.401σ√T</td>
<td>1.801σ√T</td>
<td>1.285</td>
</tr>
</tbody>
</table>

Note that so far we have just stuck with the Normal distribution and have not allowed jumps, or stochastic volatility.

Various jump models have been proposed, but it is sufficient for our purposes just to review the empirical results of three of those:

- Merton (1976)
- Carr, Geman, Madan, Yor (2002)
- Carr and Wu (2003)

Jump models fall into two distinct groups; those that allow
finite jumps (i.e. infrequent), and those that are called infinite jumps (i.e. lots)

As an aside, there is strong evidence for predictability of jumps, at least in the US markets. It seems that the VIX index can predict both jump arrival (timing) and size.10 There is also evidence of jump clustering, and as a result we would tend to expect that the Merton (1976) model would not be as useful as the others since it cannot capture persistence in the data – it has no memory. However, at least empirically all three models discussed here seem to fit the data equally well – or at least not statistically significantly differently. The results are repeated from the Bakshi and Panayatov paper. (see table below)

These multiples are compared to standard Normal VaR:
- JD = Merton’s jump-diffusion model
- CGMY is the two-sided pure-jump Levy model of Carr, Geman, Madan, and Yor
- FMLS is the finite-moment log-stable model of Carr and Wu

We can clearly see that
1. VaR with jumps is bigger than standard Normal VaR, although no doubt we suspected as much beforehand…

2. The choice of model makes a difference, although they all fit the data equally well. The FMLS model of Carr and Wu allows the fattest tailed distributions, and hence the most dramatic multipliers.

3. VaR-I is consistently greater than VaR, which we suspected based on the known result with the Normal, no drift case.

4. VaR-I can be more than double standard VaR, and this goes some way toward justifying the Basel multipliers.

At this point we have relaxed the Normal distribution assumption, allowed drift, and allowed for jumps in the return process under various different distributions. However, of course there are more things that we could try. Here are a few pointers for further reading:

2. Two-dimensional PIDEs for models with both jumps and stochastic volatility - Feng and Linetsky (2006)
3. Leaning heavily once again on the barrier-option analogy we could also consider:
   - A risk range rather than a single loss level (double barrier)
   - A time-varying barrier or range
   - A barrier that varies as a function of volatility
   - WwEg. Lo & Hui (2007), pricing double barriers where volatility, dividend yield, and the barriers are stochastic.

To summarize, intra-horizon risk is an important and neglected topic. If you are a buy-side portfolio manager you can probably implement and use Normal distribution formulation for first passage probability with drift given earlier in this note, since the central limit theorem has a surprisingly rapid effect, and even fairly concentrated portfolios rapidly converge toward Normal distributions over all but the highest frequency.11 If you are a levered investor or a trading desk, then you will most likely need to resort to more flexible processes that allow fat tails and jumps in returns.

**End Notes**
3. For evidence for jumps, see Bakshi, Cao, Chen (1997), Bates (2000), Anderson, Benzoni, Lund (2002). There is also evidence for large jump risk premia; Pan (2002)... but difficulty explaining market crashes with return jumps and diffusive volatility... For evidence for jumps in return and in volatility see Eraker, Johannes, Polson (2003). For two examples of models with return and volatility jumps see Duffie, Pan, Singleton (2000)
4. This is a largely overlooked topic. There are a few good papers: Stultz (1996), Kritzman and Rich (2002), Boudoukh, Richardson, Stanton, Whitelaw (2004). However, these papers make restrictive assumptions: Brownian motion, no jumps, stationary volatility etc. One paper addresses the distributional issues and includes results for processes with jumps: Bakshi, Panayatov (forthcoming JFE)
5. See for example Karatzas and Shreve (1991)
7. If you do resort to MC approaches, please take the time to read the excellent paper by Atiya & Metwally (2002) – they present a very fast approach, around 100 times faster than pure MC by leveraging a “Brownian bridge” to reduce the number of points that need to be calculated
8. Feller (1971), 1% level.
References

- Carr, Geman, Madan, Yor, 2002.
- Merton, Reiner, Rubenstein, 1973
- Merton, 1976